Hanner, and Karhunen had shown that, (in Cramèr's terminology) if ξ is a (scalar) stationary process which is purely nondeterministic then it has multiplicity $M = 1$ and spectral type $([m])$, where m is Lebesgue measure. This contrasts with the results for nonstationary processes where, even in the purely nondeterministic case any value of M can occur.

In the book under review Rozanov surveys the indicated problem area, including the situation where ξ_t may be vector space valued. Rozanov himself has made many contributions toward the solutions of these problems. It seems remarkable that he manages to give complete proofs and numerous examples in this book of 133 short pages. The translation from the Russian, edited by A. V. Balakrishnan, reads very well. The book should be welcome by both novice and experts in the field.

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Equations of mixed type, by M. M. Smirnov, Transl. Math. Monographs, Vol. 51, American Mathematical Society, Providence, R. I., 1978, iii + 232 pp., $27.20.

Just what is an equation of mixed type? Equation here means partial differential equation, and if some of these are of mixed type, there must be others not of mixed type. What are they? To answer these questions we must know the labels which are attached to various classes of partial differential equations. As one would expect, the labeling process has evolved over the years in a disorderly way; by now however the terminology has stabilized for many (but far from all) classes of equations. As is the case for the problem of taxonomy in the biological sciences, the subdivision of partial differential equations into clearly defined classes has not been systematic. New terms continue to develop as the need arises. For example, the term strongly elliptic was invented to identify a special subclass of the class of elliptic equations. Classes overlap: hypoelliptic equations contain some elliptic equations and some which are not elliptic; both the class of linear equations and the class of nonlinear equations contain elliptic and nonelliptic subclasses. Someday it may be worthwhile to try to systematize completely the taxonomy of partial differential equations. Meanwhile, in order to identify the niche occupied by equations of mixed type, we give here a brief account of where we now are in the classification process.

Let D be a region in R^n, $n \geqslant 2$, and u a smooth function from D into R^1. We denote by $x = (x_1, x_2, \ldots, x_n)$ an element of R^n. With $\alpha_1, \alpha_2, \ldots, \alpha_k$ positive integers and $D^{\alpha_j}u$ a partial derivative of u of order α_j with respect to any combination of the x_i, a partial differential equation is an equation of the form

$$F(x, u, D^{\alpha_1}u, \ldots, D^{\alpha_k}u) = 0. \tag{1}$$

The *order* of such an equation is the largest of the integers $\alpha_1, \ldots, \alpha_k$.

One of the most important problems of partial differential equations, designated the Basic Problem, is easily stated: suppose that some information about u (e.g. the value either of u, or some of its derivatives, or other quantities involving u and the derivatives of u) is known on all or part of the boundary of D. When does there exist a unique solution u of (1) in D which satisfies the conditions prescribed on ∂D and which varies smoothly with these conditions? Since the answer may depend on the particular geometry of D as well as on the information given on ∂D, an additional facet of the Basic Problem is the determination of those regions D and those conditions on ∂D for which a unique, smoothly varying solution exists.

The assumptions on the function F are crucial for solvability. The simpler the form of F, the easier it is to attack the Basic Problem. As a result, the study of partial differential equations has been fractured into large numbers of classes, each consisting of a collection of functions F with the property that progress on the Basic Problem can be made for that particular set of functions. The order of the equation provides one easy decomposition into classes; in particular, equations of the second order, especially those related to problems in mathematical physics and engineering, have been studied most intensively and for the longest time.

A partial differential equation is *linear* if F is a linear form in u and its partial derivatives. The division of equations into linear and nonlinear classes is a natural one since the methods for attacking the Basic Problem are quite different for these two classes. The classifications, certainly bewildering to the nonspecialist, continue. For example, nonlinear equations may be semilinear, or fully nonlinear with each of these subclasses decomposed further into equations having special properties. Linear equations may be homogeneous, nonhomogeneous, with constant coefficients, contain terms of only one order, and so forth.

The Basic Problem has been solved for some small classes of equations. However, for most linear equations practically nothing is known about the solution of the Basic Problem.

A classification of equations has developed which is related to the geometry of the region D and the kinds of conditions prescribed on the boundary of D. By this means an equation is identified according to its *type*. In describing the type of an equation we shall limit ourselves to linear equations of the second order, although similar definitions can be made for equations of any order, both linear and nonlinear. In fact, systems of partial differential equations (i.e., equations such as (1) in which both u and F are vector-valued) have also been classified in various ways, but the classifications of systems are much less comprehensive than those of a single equation. We put aside entirely the discussion of the decomposition into classes of systems of partial differential equations.

The general second order linear partial differential equation

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x), \; a_{ij}(x) = a_{ji}(x) \qquad (2)$$

is of *elliptic type at a point* x if the quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$$

is never zero for all $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ in R^n except $\xi = 0$. That is, an equation is elliptic if the signature of the quadratic form consists of n plus signs (or n minus signs). The equation is elliptic in a region D if it is elliptic at every point of D. If at a point x the signature of the quadratic form consists of $n - 1$ plus signs and one minus sign, the equation is of *hyperbolic type at* x. If the rank of the form is $n - 1$ and the signature has $n - 1$ plus signs and if the $b_i(x)$, $i = 1, \ldots, n$, are restricted properly, then the equation is of *parabolic type at* x. In general the *type at a point* x of an equation (2) is described by the rank and signature of the quadratic form, although in some cases properties of the first order terms in (2) are needed.

Most of the results on the Basic Problem in partial differential equations have been obtained for those equations which are elliptic or parabolic or hyperbolic throughout a region. Although other types of equations have been identified, we know very little about them. For example, equations of *ultrahyperbolic type* are those for which the signature of the quadratic form consists of k plus signs and $n - k$ minus signs with $n \geqslant 4$ and $1 < k < n - 1$. In contrast with the rich theory which has developed for elliptic, hyperbolic, and parabolic equations, not a single solution to the Basic Problem has been obtained for an ultrahyperbolic equation in any region D for any given set of conditions on all or part of ∂D, despite the fact that ultrahyperbolic equations have been studied for some time. Moreover, if the signature of the quadratic form has plus and minus signs and is anything other than those described above, not only is very little known about the solution to the Basic Problem but there is not even a generally accepted nomenclature which identifies equations according to their type.

The type of an equation may change from point to point. For example, if the form $\Sigma_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$ is nonnegative for all values of $\xi \neq 0$ and all x in D, then (2) may be elliptic at some points, parabolic at others, and have rank less than $n - 1$ (even zero) at still other points. The term equations with *nonnegative characteristic form* is used to identify this class. More generally, we say that equation (2) is of *variable type in a region* D if (2) is not of one type throughout D. Sometimes hyphenated terms such as elliptic-parabolic, elliptic-hyperbolic and hyperbolic-parabolic are used to specify more precisely the mixture of types. Equations of nonnegative characteristic form contain as a subclass those of elliptic-parabolic type. The term equations of mixed type has developed a special meaning among equations of variable type. Suppose that D consists of two subregions D_1 and D_2 with a common boundary Γ. If an equation is elliptic in D_1, hyperbolic in D_2 the equation is said to be of *mixed type in D*. This definition is by no means universal with the word mixed used in many contexts. For example, some authors use the term mixed hyperbolic-parabolic type and others use the word mixed for any equation of variable type.

Some progress has been made on the Basic Problem for certain classes of equations of mixed type, with most of the work being done for linear second order equations in R^2. For equations in R^2, the classification problem is simplified a great deal with (2) taking the form

$$a_{11}(x)\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}(x)\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}(x)\frac{\partial^2 u}{\partial x_2^2} + b_1(x)\frac{\partial u}{\partial x_1} + b_2(x)\frac{\partial u}{\partial x_2} + c(x)u = f(x).$$

This equation is elliptic, hyperbolic, or parabolic according as $a_{11}a_{22} - a_{12}^2$ is positive, negative, or zero.

In 1923 Tricomi initiated the work on equations of mixed type when he solved the Basic Problem for the equation

$$x_2\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \tag{3}$$

(now known as the Tricomi equation) for a special region D in which (3) is of mixed type. For about 20 years the work of Tricomi drew little attention and only a few papers appeared extending his results to somewhat more general equations and regions. The connection of the work of Tricomi to problems in transonic flow was recognized in the 1940s; since then there has been a steady stream of results of all kinds for equations of mixed type and related equations of variable type.

The book of Smirnov under review is devoted primarily to a discussion of the Basic Problem for the equation

$$K(x_2)\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + a(x)\frac{\partial u}{\partial x_1} + b(x)\frac{\partial u}{\partial x_2} + c(x)u = f(x) \tag{4}$$

where K is a continuous strictly increasing function of x_2 with $K(0) = 0$. (4) is elliptic, hyperbolic, or parabolic according as x_2 is greater than, less than, or equal to zero. For a variety of special regions D in which (4) is of elliptic-hyperbolic type, Smirnov presents detailed proofs of theorems of existence, uniqueness and smooth dependence on the boundary data for several kinds of conditions on part of the boundary of D.

Smirnov's book was written about ten years ago; if viewed from the perspective of solving the Basic Problem for the entire class of equations of variable type, its results seem rather special, as indeed they are. Nevertheless the book is a thorough compilation of what was known up to that time about the Basic Problem for (4).

Since the book was written there has been considerable activity in equations of variable type, especially in the Soviet Union. As one might expect in a newly developing field, many of the results are fragmentary. For example, a spate of papers has dealt with equations with discontinuous coefficients. Other papers have developed properties of solutions of variable equations of order higher than two. Still a third group of results is based on the equations of gas dynamics which are of mixed type when the flow has both supersonic and subsonic regions.

It seems likely that the bits and pieces of all these results will be put together eventually to form a single comprehensive theory for mixed type equations in R^2 similar to the classical theory for the standard elliptic, hyperbolic and parabolic equations. At this time the task appears formidable; the development in three and more dimensions and for equations of order other than two is even more remote.

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The Bochner integral, by Jan Mikusiński, Academic Press, New York, 1978, xii + 233 pp.

In the 1930's a great effort was made to develop the basic theory of Banach space valued functions of a real variable. The pioneers in this study (Bochner, Dunford, Gelfand, Pettis, Phillips and Richart) developed a number of integrals of varying strengths for a multitude of purposes: oftentimes, the representation of operators on concrete spaces was the object; quite as often a desire just to understand the abstract process of integration was sufficient motivation. Of the integrals developed, one integral, the Bochner integral, emerged as the strongest and, to-date, it is the Bochner integral that has been the most useful.

Curiously, the Bochner integral is the easiest of the vector integrals from yesteryear to develop and the one with the most transparent structure. Indeed, most of the usual results valid for the Lebesgue integral easily adapt to the Bochner setting. One notable exception: the fundamental theorem of calculus for absolutely continuous functions defined on [0, 1]. It is simply not the case that an absolutely continuous vector-valued function defined on [0, 1] need be the indefinite Bochner integral of its derivative–at least not unless the vector values are suitable chosen. This pathology is not all together a bad thing. The study of the class of Banach spaces for which the fundamental theorem remains valid has kept a number of mathematicians busy and off the streets for the past five years at least. This class of spaces (whose members answer to the name "Radon-Nikodym") has come to play an important role in modern Banach space theory especially as it interacts (and it does so quite nicely) with probability theory, harmonic analysis and the infinite dimensional topology. Any book purporting to be about the Bochner