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HOMOLOGY STABILITY OF GL_n OF A DEDEKIND DOMAIN[1]

BY RUTH M. CHARNEY

The purpose of this paper is to prove that for a Dedekind domain Λ, the homomorphisms

$$H_i(GL_n(\Lambda); \mathbf{Z}) \longrightarrow H_i(GL_{n+1}(\Lambda); \mathbf{Z})$$

are isomorphisms for n sufficiently large. This problem has been of particular interest to K-theorists since $K_i(\Lambda) = \Pi_i(BGL^+(\Lambda))$ where $BGL^+(\Lambda)$ is a topological space with the property that $H_*(BGL^+(\Lambda)) \cong H_*(GL(\Lambda)) \cong \varinjlim H_*(GL_n(\Lambda))$. In particular, if Λ is a ring of algebraic integers in a number field, then the groups $GL_n(\Lambda)$ are algebraic groups, and the stability theorem allows us to apply the well-developed theory of algebraic groups toward computations of K-groups. For example, Borel-Serre [2] show that for these rings, $GL_n(\Lambda)$ has finitely generated homology. Stability then implies that $H_i(Gl(\Lambda))$ and hence $K_i(\Lambda)$ is finitely generated, thus giving a new proof of a theorem of Quillen [5]. In addition, for $\Lambda = \mathbf{Z}$, a good deal is known about p-torsion in $GL_n(\mathbf{Z})$. It is conjectured that this will give p-torsion information about $H_*(GL_n(\mathbf{Z}))$ and hence, via the results of this paper, about $K_i(\mathbf{Z})$. (At the moment, the connection between torsion in a group and torsion in the homology of the group is not entirely understood, but considerable progress in this direction has been made by K. S. Brown [4] and C. Soulé [8].)

The exact statements of the main theorems are as follows:

THEOREM 1. *For V, W finitely generated projective modules over a Dedekind domain,*

(i) $H_k(\mathrm{Aut}(W \oplus V), \mathrm{Aut}(W); \mathbf{Z}) = 0$ *for* $\mathrm{rk}\ W \geq 4k + 1$,

(ii) $H_k(\mathrm{Aut}(W \oplus V), \mathrm{Aut}(W); \mathbf{Z}[½]) = 0$ *for* $\mathrm{rk}\ W \geq 3k + 1$.

THEOREM 2. *For Λ a PID, $G_n = GL_n(\Lambda)$ or $SL_n(\Lambda)$*

(i) $H_k(G_{n+1}(\Lambda), G_n(\Lambda); \mathbf{Z}) = 0$ *for* $n \geq 3k$,

(ii) $H_k(G_{n+1}(\Lambda), G_n(\Lambda); \mathbf{Z}[½]) = 0$ *for* $n \geq 2k$.

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THEOREM 3. *For* Λ *a PID such that* $SK_1(\Lambda) = 0$,

(i) $\Pi_k(BGL_{n+1}^+(\Lambda), BGL_n^+(\Lambda) = 0$ *for* $n \geq 3k$,

(ii) $\Pi_k(BGL_{n+1}^+(\Lambda), BGL_n^+(\Lambda)) \otimes \mathbf{Z}[1/2] = 0$ *for* $n \geq \max(2k, 3)$.

Similar theorems have been obtained by Quillen and Wagoner for Λ a field or local ring, and by Vogtmann, Friedlander, and Alperin for various other classical groups (see [9], [10], and K. S. Brown's survey article in [3]). The only such result which applies to $\Lambda = \mathbf{Z}$ is Borel's theorem for $H^*(SL_n(\Lambda); \mathbf{Q})$, Λ a ring of algebraic integers [1].

In this paper, we follow the approach used by Quillen in [6] and Vogtmann in [9]. Namely, we construct a simplicial complex with the homotopy type of a wedge of spheres and use the action of GL_n on this complex to prove our results.

Recall that the Tits building $\boxed{V}$ on a vector space V is the geometric realization of the partially ordered set of proper submodules of V. If $\dim V = n$, Solomon and Tits [7] have shown that $\boxed{V} \simeq VS^{n-2}$.

The action of GL_n on $\boxed{\Lambda^n}$ gives rise to a spectral sequence converging to 0 with

$$E'_{p,q} = H_q\left(\begin{array}{|c|c|}\hline GL_p & * \\ \hline 0 & GL_{n-p} \\ \hline\end{array}\ ; H_{p-2}\left(\boxed{\Lambda^p}\right)\right)$$

where the groups

$$\begin{array}{|c|c|}\hline GL_p & * \\ \hline 0 & GL_{n-p} \\ \hline\end{array}$$

arise as the stabilizer of the vertex Λ^p in $\boxed{\Lambda^n}$.

Our problem in this paper is to obtain a spectral sequence whose E' terms involve matrix groups of the form

$$\begin{array}{|c|c|}\hline GL_p & 0 \\ \hline 0 & GL_{n-p} \\ \hline\end{array}$$

rather than

$$\begin{array}{|c|c|}\hline GL_p & * \\ \hline 0 & GL_{n-p} \\ \hline\end{array}$$

This allows us to use the Kunneth formula to reduce to lower dimensional cases and apply induction. With this in mind, we define a "split building" on a module W whose vertices are pairs (P, Q) of proper submodules such that $P \oplus Q = W$. We denote this complex by $[W]$. Though it is not, in the technical sense, a building, we prove that if W is a rank n projective module over a Dedekind domain, then $[W] \simeq VS^{n-2}$.

The proof of this is geometric. We begin with a contractible subcomplex

$$X_0 = \text{subcomplex of vertices} \leqslant \text{a fixed vertex } (H, L),$$

and construct $[W]$ by successively adding the remaining vertices. Adding a new vertex (P, Q) consists of attaching a cone to the link of (P, Q) with the previously constructed subcomplex. We use induction on rank W to show that this link is homotopic to VS^{n-3} so that the resulting subcomplex is homotopic to VS^{n-2}.

We next define a filtration

$$[W]_1 \subset [W]_2 \subset \cdots \subset [W]_{n-1} = [W]$$

and prove in a similar manner that $[W]_j \simeq VS^{j-1}$. The spectral sequence for this filtration gives an exact sequence of Aut(W)-modules

$$0 \leftarrow \mathbf{Z} \leftarrow H_0([W]_1) \leftarrow H_1([W]_2, [W]_1) \leftarrow \cdots$$
$$\leftarrow H_{n-2}([W]_{n-1}, [W]_{n-2}) \leftarrow H_{n-2}([W]) \leftarrow 0.$$

Denoting this sequence by $C_*(W)$ and letting $E_*\text{Aut}(W)$ be a resolution of $\mathbf{Z}$ by free Aut(W)-modules, we show that the spectral sequence associated with the double complex $E_*\text{Aut}(W) \otimes C_*(W)$ converges to 0 and has

$$E'_{p,q} = \Sigma\, H_p(\text{Aut}(A) \times \text{Aut}(B); H_{p-2}([A]))$$

where the sum ranges over isomorphism classes of vertices (A, B) in $[W]$ such that rank $A = p$. The proof of Theorem 1 is now completed by comparing this spectral sequence with the corresponding one for $W \oplus V$. Restricting to PID's, the spectral sequence simplifies, and a similar argument gives Theorem 2. Theorem 3 follows as an immediate corollary to Theorem 2 when we observe that the hypothesis $SK_1(\Lambda) = 0$ implies that $BSL_n^+(\Lambda)$ is the universal covering space of $BGL_n^+(\Lambda)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720