short discussion of normed vector lattices leads to the L^p spaces and the abstract L- and M-spaces. Kakutani's concrete representation of L-spaces is given, but strangely enough the corresponding one for M-spaces does not appear.

A novel topic appears in Chapter 7–linear spaces with norm-functions that are vector-lattice valued. The results are reminiscent of those in vector-valued function spaces. In particular, integral representations are obtained for bounded linear operators on the space of continuous vector-valued functions and on the space of (Bochner) integrable vector-valued functions (where the domain of these functions is a compact interval on the real line). The Hellinger integral is developed for this last purpose. Finally, integration of vector-valued functions with respect to vector measures (via a bilinear map) is presented.

The last chapter "... gives a brief exposition of the manner in which the theory of ordered vector spaces can be used in various branches of mathematics." These include operator equations in various contexts, operator extensions, the spectral theorem for selfadjoint operators on Hilbert space, and fixed points for positive contractions.

The book is well organized and clearly written. The level of exposition is detailed, yet important material is easily accessible (if one already knows what's important–there are no real indications of the high points). Each chapter ends with bibliographic notes which reference and complement the text material. The biggest drawback of the book is that there are no exercises or problems whatever, and very few examples. In fact, the whole circle of motivating examples available in Orlicz spaces, Banach function spaces, normed Köthe spaces is not mentioned at all. In addition, with the exception of work by the author, the bibliography stops at 1970 and omits the three major recent books on the subject, viz., G. Jameson's *Ordered linear spaces* (1970), A. L. Peressini's *Ordered topological vector spaces* (1967) and the first volume of the very substantial and important *Riesz spaces* (1971) by W. A. J. Luxemburg and A. C. Zaanen.

Nonetheless, the book includes a good selection of material organized in a usable fashion and would make a good reference and text, if properly supplemented.

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Subharmonic functions, by W. K. Hayman, and the late P. B. Kennedy, Vol. I, Academic Press, London, New York, San Francisco, 1976, xvii + 284 pp., $25.50.

Subharmonic functions have been around for a long time, although not known by that name originally, and have played a central role in the development of mathematics. The Newton and Coulomb inverse square laws for gravitational and electromagnetic forces, respectively, made this role inevitable. These laws were used to define potential functions representing the potential energy of a unit mass or charge at a point of three dimensional Euclidean space due to a nonnegative mass or charge distribution on or within a closed surface not containing the point. The negatives of such functions are now known as subharmonic functions.

During the initial period of development, up to the middle of the nineteenth century, potential theory was more physics than mathematics and reasonable physical arguments were used. Near the end of the eighteenth century, Laplace had shown that a potential function satisfies a partial differential equation, which now bears his name, at all points outside the support of the mass distribution, and, around 1820, Poisson found a representation of a function satisfying Laplace's equation on a neighborhood of a closed ball in terms of an integral of the function on the boundary of the ball, the familiar Poisson integral formula. Between 1820 and 1830, both G. Green and K. F. Gauss investigated the problem of determining a potential function which is constant within and on a closed surface, satisfies Laplace's equation at all points outside the surface, and, if a boundary condition corresponding to a grounded conductor is imposed, within a second surface. Green approached the problem by constructing a kernel, the construction of which amounted to showing that a prescribed function on a closed surface could be extended to the interior of the surface and satisfies Laplace's equation therein. The kernel constructed in this way has since become known as "the Green function" or "Green's function" (or even "the Green's function" by some). Green had no doubts about the existence of such a kernel because of the physical situation. Gauss proceeded somewhat differently by arguing that the sought for potential function was the potential energy of an equilibrium distribution on the surface and must minimize a functional, now known as Gauss' integral, of mass distributions. Gauss too had no doubts about the existence of such an equilibrium distribution. As an aside, it was Gauss who introduced the logarithmic potential in connection with Laplace's equation for planar regions.

By the middle of the nineteenth century, serious questions had been raised about the validity of the physical arguments used by Green, Gauss, and others, principally by K. Weierstrass who showed that the type of variational argument used by Gauss need not be valid. The problem of showing that a continuous function on a closed surface has a continuous extension to the interior of the surface which satisfies Laplace's equation therein had become known as the Dirichlet problem. It was in this period that B. Riemann introduced a variational principle, known as the Dirichlet principle, according to which the solution of the Dirichlet problem is the function on a prescribed region which minimizes the Dirichlet or energy integral subject to the condition that the function takes on prescribed boundary values. Riemann repeated the errors of Green and Gauss but in a different formulation. By 1870, mathematicians had four closely related problems on their hands dealing with the Dirichlet problem, the equilibrium distribution, the Green function, and the Dirichlet principle. An interesting account of the early history of potential theory can be found in [2]. O. D. Kellogg's book [1] also contains much interesting history.

During the last two decades of the nineteenth century, the Dirichlet problem was shown to be solvable, first for regions with sufficiently smooth boundaries by Carl Neumann, then for special regions by H. A. Schwarz using the "alternating method" which bears his name, and then by H. Poincaré for regions having the property that each boundary point lies on a sphere which does not contain any interior points of the region (according to Monna [2], Poincaré gave the name "harmonic" to functions satisfying Laplace's equation). Shortly after the turn of the century, D. Hilbert succeeded in justifying the Dirichlet-Riemann principle for sufficiently nice regions and boundary functions and W. F. Osgood proved the existence of the Green function for fairly general simply connected planar regions. Although the contributions of Neumann, Schwarz and Poincaré established the solvability of Dirichlet's problem for some regions, it was not until 1911 that Zaremba gave the first example of a region for which Dirichlet's problem is not solvable, namely a punctured disk in the plane with a boundary function equal to 1 at the deleted point and zero on the circumference. In 1913, H. Lebesgue constructed a counterexample in three dimensions. Lebesgue's example simultaneously showed that the Dirichlet problem is not solvable and that the physical arguments used by Green were not valid. In 1915, G. H. Hardy proved that the logarithm of the modulus of an analytic function satisfies a subaveraging principle according to which its value at a point is dominated by its average over the boundary of a closed disk centered at the point and contained in the domain of the function; this result was the forerunner of the results in the book under review. O. Perron in 1923 and N. Wiener in 1924 each introduced new methods of solving the Dirichlet problem which in effect modified the Dirichlet problem by associating with each boundary function a related harmonic function; the methods were equally applicable to discontinuous boundary functions. As a result, a new problem arose in showing that the harmonic function so constructed has the proper limit at points of the boundary. Using the concept of capacity, which was introduced into mathematics by N. Wiener for compact sets, G. C. Evans showed in the thirties that the harmonic function so constructed had the correct limiting values at continuity points of the boundary function with the possible exception of a set of outer capacity zero. But new concepts lead to new problems; capacities of compact sets lead to inner and outer capacities for arbitrary sets and to the question of what sets are capacitable (that is, have equal inner and outer capacities).

The use of the term "subharmonic function" commenced in 1926 when R. Riesz defined a real-valued function on an open set to be subharmonic if it does not take on the value $+\infty$, is upper semicontinuous, and satisfies the subaveraging principle. Among the examples of subharmonic functions given by Riesz are the functions $\log|f(z)|$ where f is analytic.

The thirties brought about the final resolution of many of the classical problems in potential theory. In 1935, O. Frostman justified the existence of a measure minimizing Gauss' integral. In 1933, P. J. Myrberg characterized those regions having a Green function as those supporting a nonconstant positive superharmonic function or, alternatively, as those for which the complement is not the set of infinities of a nonconstant superharmonic function. Also in the thirties the Dirichlet principle was put on a sound basis by Zaremba and Nikodym with subsequent improvements made by Brelot, Weyl, and Deny. A general theory of capacities was developed by G. Choquet in 1954 in which it was established that each analytic set is capacitable. An important new step was taken in 1956 when M. Brelot recognized that a theory of harmonic functions could be built on just a few properties, thereby freeing potential theory from a particular partial differential equation, Laplace's equation, and making most of the previously developed theory applicable to elliptic partial differential equations. This axiomatization of potential theory was soon followed by an axiomatization of solutions of the heat equation by H. Bauer.

Most of the above developments are included in the book under review with most illustrative examples coming from complex function theory. The first and last thirds of the book cover most of the standard topics in potential theory, the Green function, the Poisson integral formula, the maximum principle, the Dirichlet problem, capacities, and negligible sets. The middle third distinguishes this book from other potential theory books in that it deals mostly with the growth of subharmonic functions at infinity. This topic is an outgrowth of work on the particular subharmonic functions $\log|f(z)|$ by R. Nevanlinna, M. Heins, L. V. Ahlfors, and others since 1929 and falls under the rubric of Nevanlinna theory. The basis of this theory is found in F. Riesz' 1926 paper in which it is proved that if u is subharmonic on R^n and E is an open set with compact closure, then there is a unique Borel measure μ on R^n such that u can be decomposed as the sum of the potential of μ and a harmonic function on E. The growth of u at infinity is investigated by comparing the supremum of u on a sphere of radius r and center at the origin with the average of u^+ over the same sphere, by comparing the μ measures of balls of radius less than or equal to r with the supremum of u over the sphere of radius r, and by comparing the average of u^- with the average of u^+ over a sphere of radius r. Such comparisons were extensively studied in the thirties in the $\log|f(z)|$ case and the results of these studies have been extended to subharmonic functions on R^n by the authors, B. Dahlberg, and several others. Considerable attention is also given to generalizations of a classical theorem of Iverson which states that if $f(z)$ is a nonconstant entire function, then $f(z) \to \infty$ as $z \to \infty$ along some path.

Professor Hayman states in Acknowledgements that he hopes the book will serve as a memorial to his friend and former student Professor P. B. Kennedy who died in 1967. The book is thorough, well written, and will surely serve as a fitting memorial.

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