perfect rings. However, this latter book does not discuss Q.F. rings, but gives instead a broad discussion of Morita equivalence and Morita duality. So in spite of a large overlap, the aims of the books are still different.

I close the review with expressing the hope that the fact that the book under review is written in German won't frighten too many prospective readers away.

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Hopf spaces, by Alexander Zabrodsky, Mathematics Studies, No. 22, North-Holland, Amsterdam, 1976, x + 223 pp., $18.50.

The subject of H-spaces is generally agreed to have begun in 1941 with the publication of Hopf's paper [5]. In the proceedings of the 1970 Neuchâtel conference on H-spaces, James [8] listed 347 entries for a bibliography on H-spaces. Since that time numerous articles on H-spaces have been published. It is therefore somewhat surprising that Zabrodsky's monograph is only the second book to appear which deals with H-spaces in general. Before discussing the book, I would like to provide some background on the subject itself.

It is quite easy to define the basic concept. An H-space (or Hopf space) consists of a topological space X with chosen point $* \in X$ and a continuous function μ: $X \times X \to X$ called the multiplication or H-structure on X. The requirement is that $*$ be a two-sided unit up to homotopy, that is, the maps $x \to \mu(x, *)$, $x \to \mu(*, x)$, and the identity map of X are all to be homotopic. If in the definition we replace homotopy by equality and write $\mu(x, y)$ as $x \cdot y$, we obtain $x \cdot * = * \cdot x = x$. The multiplication is then called strict and we shall refer to the resulting object as a topological quasi-group, a precursor of a topological group. Two important classes of examples of H-spaces are topological groups and the space of loops ΩY of an arbitrary space Y. The latter consists of continuous paths in Y parametrized by [0, 1] which begin and end at a fixed point of Y with multiplication of paths the same as in the definition of the fundamental group. H-spaces are studied because they are a natural object in homotopy theory and because they are a unifying concept for several diverse examples. Moreover, the H-space conditions are often sufficient to prove certain results about topological groups. An example of this is the following well known theorem of Hopf [5] which applies to compact Lie groups: the cohomology algebra $H^*(X; F)$ of a finite H-complex (i.e., an H-space which is a finite complex) with coefficients in a field F of characteristic zero is isomorphic to an exterior algebra on odd dimensional generators.

Although the subject has no single theorem or collection of techniques as a focal point, for purposes of discussion it is convenient to subdivide most of it into three major areas: (1) algebraic topology of H-spaces, (2) group theory of H-spaces, and (3) classifying spaces and iterated loop spaces.

The first area is concerned with the homotopy groups, the homology groups, the cohomology algebra, the cohomology operations, the K-theory, etc. of H-spaces. What restrictions does the existence of the multiplication μ place on the algebraic invariants of X? The answer is often in terms of additional algebraic structure. On the homotopy groups $\pi_*(X)$, for example, μ induces a binary operation called the Samelson product which converts $\pi_*(X)$ into a graded Lie ring. In homology $H_*(X; R)$ with coefficients in a ring R μ determines the Pontryagin product, defined as the composition

$$H_*(X; R) \otimes H_*(X; R) \to H_*(X \times X; R) \xrightarrow{\mu_*} H_*(X; R),$$

which makes $H_*(X; R)$ into a graded ring. In fact, it is the Pontryagin product and the homology homomorphism induced by the diagonal map $X \to X \times X$ which provide $H_*(X; F)$ with the structure of a Hopf algebra over a field F. Dually, the cohomology $H^*(X; F)$ is a Hopf algebra. The theorem of Hopf cited above and its extension by Borel [12] to perfect fields F, give the multiplicative and additive structure of this Hopf algebra. Results of this kind can be used in two ways: either to obtain information about the algebraic invariants of a given H-space or to conclude that certain spaces cannot be H-spaces. The most striking example of this latter kind of negative result is the famous theorem of Adams [1] that the only spheres which admit H-structure are S^0, S^1, S^3, and S^7.

In the second major area one exploits the analogy between H-spaces and quasi-groups to investigate the homotopy version of quasi-group theoretic problems For instance, when is an H-space X homotopy-associative, i.e., when are the two maps $(x, y, z) \to (x \cdot y) \cdot z$ and $(x, y, z) \to x \cdot (y \cdot z)$ of $X \times X \times X$ to X homotopic, when do homotopy-inverses exist, and when is X homotopy-commutative? One may compare different multiplications on the same space and attempt to enumerate them up to homotopy. The appropriate homotopy notions of homomorphism (H-map) and isomorphism (H-equivalence) of H-spaces are studied. For an H-space which satisfies all of the group axioms up to homotopy (e.g., a loop space), it is natural to raise questions about homotopical nilpotency and homotopical solvability. The algebraic methods described in the previous paragraph clearly play a crucial role in these considerations. As an example of the results in this area we mention an interesting theorem due to Hubbuck [6]: every homotopy-commutative finite H-complex has the homotopy type of a point or a product of circles.

The third area deals with the relationship between H-spaces and loop spaces. If X is a topological group, the classifying space B_X of X [11] has the property that the loop space ΩB_X is equivalent to X. It is the associativity of the topological group that allows this, and one investigates à la Stasheff [14] the relationship between higher homotopy-associativity conditions on an H-space X and the existence of approximations to a classifying space of X. If X is equivalent to the loop space ΩX_1, one may ask if X_1 is equivalent to some ΩX_2, and so on. Thus there is the problem of recognizing when an H-space is equivalent to an n-fold iterated loop space, $n < \infty$. Of special interest are the H-spaces which arise in the stable classification of fibrations, and which turn out to be infinite loop spaces. There are many homological questions here such as the cohomology relationship between a topological group and its classifying space, between a space and its iterated loop space, and homology operations for iterated loop spaces.

In addition to these three areas, a fourth has appeared in recent years. This concerns the theory of H-spaces modulo a prime p. Techniques such as localization enable one to associate to a space and a prime p an auxiliary space which captures the mod p homology and homotopy of the given space. This approach has been successfully applied to H-spaces and has given rise to a whole slew of new finite H-complexes with interesting properties.

It should be clear from the preceding discussion that the theory of H-spaces both influences and is influenced by other parts of topology and algebra such as Lie rings, Hopf algebras, Lie groups, classifying spaces, fibrations, and fibre bundles.

The existing books and lecture notes cover the field quite well. Stasheff's book [14], the first to deal with H-spaces in general, discusses some topics in the group theory of H-spaces with emphasis on homotopy-associativity and homotopy-commutativity, and then concentrates on classifying spaces and iterated loop spaces. The monographs of Bousfield-Kan [3] and Hilton-Mislin-Roitberg [4] give much information on the mod p theory of H-spaces. A complete and detailed treatment of iterated loop-spaces appears in Boardman-Vogt [2] and May [9] (see also the review by Stasheff [15] and the survey by May [10]). Husseini [7] considers the cohomology of the classical Lie groups and their classifying spaces. The work of Milnor-Moore [12] on Hopf algebras has many applications to the homology of H-spaces.

Where does Zabrodsky's book fit into this framework? The first two chapters deal with basic concepts and are essential for an understanding of the rest of the book. They contain many known and some new results on the group theory of H-spaces. The third chapter focuses on cohomology with coefficients in $\mathbf{Z}_p$, the integers mod p. After some generalities on Hopf algebras, the author explores three topics. The first gives properties of the coalgebra structure of $H^*(\Omega Y; \mathbf{Z}_p)$ determined by relations in the algebra $H^*(Y; \mathbf{Z}_p)$. The approach is geometric and the method is illustrated by computing the coproduct of $H^*(\mathrm{Spin}\,(n); \mathbf{Z}_2)$. The second is a treatment of Browder's Bockstein spectral sequence. The theory is developed to include a sketch of the proof of Browder's first implication theorem. The last topic deals with a secondary cohomology operation of type $(\mathbf{Z}_p, 2n; \mathbf{Z}_p, 2np)$ defined for elements whose pth power is zero. For an H-space it is shown that this operation is nontrivial on primitives. In this section there is also a proof of the decomposibility of the Steenrod pth power $\mathcal{P}^p$ into primary and secondary operations. Some of Chapter III is done for a cohomology theory defined by an Ω-spectrum, but the main interest is in ordinary cohomology theory.

The fourth chapter is an exposition of the mod p theory. The author eschews the standard localization theory in favor of the p-universal approach of [13]. For a set of primes $\mathbf{P}$, a $\mathbf{P}$-equivalence is a map which induces an isomorphism of cohomology with $\mathbf{Z}_p$ coefficients for every $p \in \mathbf{P}$. A 0-equivalence is similarly defined with the field of rationals replacing $\mathbf{Z}_p$. A mod $\mathbf{P}$ H-space consists of a space X and a map $\mu: X \times X \to X$ such that $x \to \mu(x, *)$ and $x \to \mu(*, x)$ are $\mathbf{P}$-equivalences. With only minor restrictions on X the author then proves: (1) a space X is a mod $\mathbf{P}$ H-space if and only if X is $\mathbf{P}$-equivalent to an H-space (2) X is an H-space if and only if X is a mod p H-space for every prime p. The proof of (1) has not appeared previously and (2) is a new result. The latter is proved without the hypothesis that the different mod p H-structures are rationally compatible, and hence it removes a hitherto cumbersome restriction in the mod p theory. There is then a treatment of the technique of mixing due to Zabrodsky whereby 0-equivalences $Y_i \to Y_0$ and a partition $\{\mathbf{P}_i\}$ of the set of primes, $i = 1, 2, \ldots, k$, give rise to a space Y (the mixed space) which is $\mathbf{P}_i$-equivalent to Y_i. The mixing technique together with (2) enables the author to construct nonclassical finite H-complexes. One simply partitions the primes into a subset $\mathbf{P}$ and its complement $\overline{\mathbf{P}}$ and finds a known mod $\mathbf{P}$ H-space and a known mod $\overline{\mathbf{P}}$ H-space which are 0-equivalent. The resulting mixed space is then an H-space. Also included in this chapter is a discussion of p-universality, mod p homotopy, and the genus set.

In the final chapter the author advocates the use of the Brown-Peterson (BP) spectrum for handling nonstable problems. The theory is presented fairly completely and the applications are meant to be illustrative of the possibilities. The development and some of the applications seem to be new. A general geometric construction for a space is given which kills the p-torsion in homology. By applying this to Eilenberg-Mac Lane spaces $K(\mathbf{Z}, n)$, a spectrum of spaces $B(n, p)$ is obtained. A BP Adams resolution of a space Y is defined, roughly speaking, to be a sequence of principle fibrations

$$\cdots \to E_n(Y) \to \cdots \to E_1(Y) \to E_0(Y) \to E_{-1}(Y) = *,$$

each induced by maps $k_n: E_{n-1}(Y) \to \prod B(m_i, p)$, together with compatible maps $f_n: Y \to E_n(Y)$. The requirement is that f_n induces an isomorphism of homotopy groups in dimensions $< r_n$ and that $\lim r_n = \infty$. It is then indicated how to construct a BP Adams resolution for spaces Y with no p-torsion in homology such that $H^*(Y; \mathbf{Z}_p)$ is a free algebra. For an H-space, all of the spaces of the resolution can be assumed to be H-spaces and all of the maps assumed to be H-maps. Using two and three stage BP Adams resolutions, the author deduces some technical cohomological results and makes some computations of the p-primary components of $\pi_*(S^3)$ and $\pi_*(SU(n))$.

Zabrodsky's approach throughout is geometric or, more precisely, homotopy theoretic, rather than algebraic. He is adept at handling homotopies, pull-backs, Postnikov systems, liftings, homotopically defined obstructions, etc. This is evident even in the more algebraic sections. He demonstrates the advantages to be had from using strict multiplications and Moore paths, from carrying a homotopy along in a proof, and from keeping track of homotopies relative to a subspace and corelative to a fixed map. His use of pictures to describe complicated homotopies in Chapters II and III is quite ingenious. Examples and computations are interspersed throughout the book, and they serve well to illustrate the author's techniques. For instance, an upper bound is computed for the homotopical solvability of the classical Lie groups, Peterson's mod p decomposition of the classifying space B_{SU} into a product of $B(n,p)$ spaces is derived, and a lower bound on the size of the genus set of $SU(n)$ is obtained. In addition to new material, the author has incorporated much work from research articles into his book. Many known results such as James' theorem on maps from a retractile pair into an H-space are given new proofs and many familiar topics such as Browder's Bockstein spectral sequence are treated from a different standpoint.

Unfortunately the book is somewhat marred by the author's style and organization. There are numerous misprints, more annoying than confusing, notation is sometimes changed in the middle of a proof, and symbols are sometimes introduced without prior explanation. Furthermore, there are examples where the book is not as carefully organized as it could have been: Hubbuck's result is stated (without proof) twice (pp. 20 and 112), the crucial word "nonzero" is omitted in the definition of 1–implication (p. 96), the statement of Proposition 4.7.3 is confusing, the reference to a space Y_n in a paper of Wilson is nonexistent (p. 172), and the argument from the bottom of p. 121 to the top of p. 123 (case $p \notin \mathbf{P}_1$) is an immediate consequence of the preceding proposition.

The author states that he has made an attempt to bring much of the book within the grasp of graduate students. But because the prerequisites to an understanding of the book are so great, I think that the attempt has not succeeded. The prospective reader must know something about Hopf algebras, the Steenrod algebra, the Eilenberg-Moore spectral sequence, Cotor, and the BP spectrum. Even in the first two chapters, where the material is more elementary, the demands on the reader are high. Chapter 0, the chapter of preliminaries, contains only a small fraction of what is needed later on, and the references to other people's work usually allow one to see similar results proved differently.

However, these remarks are not intended to diminish the importance or the value of the book for those in the field. Zabrodsky's book is difficult and demanding, but it contains an enormous amount of both new and standard material, always presented from an interesting and original point of view. Those who work on H-spaces would do well I feel, not just to read Zabrodsky's book, but to study it and assimilate its methods. The consequences of many of the author's ideas have yet to be explored and carried further. I think that this book will influence the development of the subject for many years to come.

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Near-rings, the theory and its applications, by Gunter Pilz, Mathematics Studies, No. 23, North-Holland, Amsterdam, 1977, xiv + 393 pp., $24.00.

It has been said that the supreme occurrence in the course of an idea is that brief moment between the time it is heresy and the time it is trite. At the start of the 1960s "nonlinear functional analysis" seemed to strike most mathematicians as a contradiction in terms but by the end of that decade, some functional analysts were apologizing for considering "only the linear case."

During this period many began to realize (or to rediscover from earlier times) that quite a number of pressing scientific problems are nonlinear in nature. At the same time many others began to realize (or again, to rediscover) that many nonlinear problems have a vigorous algebraic life.

It is widely understood that many linear problems have a natural setting in some ring of linear transformations. For two illustrations (out of a vast number of possibilities) consider the following:

(1) A study of the spectrum of a bounded selfadjoint operator T on a Hilbert space H leads naturally to a consideration of the smallest closed subring of $L(H, H)$ which contains T.

(2) A strongly continuous one-parameter semigroup of bounded linear transformations on a Banach space X may be considered as a kind of ray in the ring $L(X, X)$. By now a substantial start for both nonlinear spectral