$$z'(t) \cap f(t, z(t)) \neq \emptyset,$$

where z is a linear operator on a complex Hilbert space. If Ω is the Siegel disk $z^*z \leqslant 1$, the tangent condition is often needed only on the Šilov boundary; this remark greatly increases the scope of the results. The special case

$$a(t) + b(t)z(t) + z(t)d(t) + z(t)c(t)z(t) \in z'(t)$$

applies to equations of multiple transmission lines and transport processes, and also yields results on pure operator equations (no derivatives). For example, if $b \neq 0$ and $d \neq 0$, then one of the functions

$$f(z) = a + bz(1 - cz)^{-1}d, \quad g(z) = c + dz(1 - az)^{-1}b$$

maps the Siegel disk into itself if, and only if, the other one does. Further study of operator differential equations gives results on oscillatory properties of $(pz')' + qz = 0$ which parallel those in the classical case. Extension to higher-order equations involves a far-reaching generalization of the notion of "adjoint" where, instead of an adjoint operator, one has an adjoint subspace. Among contributors to these developments are Ambartzumian, Preisendorfer, Reid, Bellman, Kalaba, Wing, Ueno, Chandrasekhar, A. Wang, Zakhar-Itkin, J. Levin, Paszkowski, Shumitzky, Helton, Krein and Shmul'yan, Etgen and Lewis, and Coddington and Dijksma.

R. M. REDHEFFER

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 4, July 1978

Nonlinear semigroups and differential equations in Banach spaces, by Viorel Barbu, Noordhoff International Publishing, Leyden, The Netherlands, 1976, 352 pp.

The typical first graduate course in ordinary differential equations begins with a discussion of the initial-value or Cauchy problem. Under a variety of assumptions, it is shown that this problem has a solution, that it is unique, and that it depends nicely on the data. Thus, under mild restrictions, Cauchy problems in classical ordinary differential equations are well posed. As the course progresses and more special topics are pursued, these preliminary results begin to seem rather simple and, in a short time, are taken for granted by the serious student. Nevertheless, one is always thinking in terms of them. Scientists and engineers often think the same way: a system being modeled has a state u which changes in time according to a differential (or evolution) equation

$$\text{(EE)} \qquad du/dt = A(u)$$

which summarizes the dynamics of the system. In classical mechanics and many other fields the state is a list of numbers (giving, e.g., velocities and positions of bodies or populations of species or quantities of reactants, etc.,) and (EE) is a classical ordinary differential equation, where "classical ordinary differential equation" means roughly that A continuously maps an open subset of some $\mathbf{R}^N$ into $\mathbf{R}^N$. One specifies an initial condition

$$\text{(IC)} \qquad u(0) = u_0$$

and studies the Cauchy problem (EE), (IC) to obtain information about the state at later times $t > 0$. The preliminary results about Cauchy problems mentioned above establish that most of the special cases of (EE), (IC) arising in these applications are well posed; however, they do not assist much with the task of making predictions from a given model.

In other applications one must deal with systems whose "states" are not lists of numbers but are instead lists of functions on a subset of some $\mathbf{R}^N$ (e.g., the velocity field in a fluid or the temperature distribution in a rod). The dynamics may still be summarized by an equation of the form (EE); but typically u now has values in some function space, and A itself is a nonlinear differential operator (or worse) subject to side or boundary conditions. The appealing geometrical interpretation of solutions of (EE) available in the classical case and the polygonal approximation (or explicit difference) method suggested by it are, in the beginning, lost to us because the A's which appear have tiny domains and are badly discontinuous. The question of well-posedness becomes interesting again because it is much less intuitive that something of this sort is true. In fact, in some examples it is no longer clear what (EE) should mean; indeed, the very problem of finding an appropriate notion of solution is quite fascinating. It is not possible that there be one universal theory for (EE), (IC) which is applicable to every model occurring in applications. However, there is an abstract framework simple enough and basic enough to make nontrivial assertions about a spectrum of problems ranging from topics in classical ordinary differential equations to problems involving the heat equation, the wave equation, the Schrödinger equation, the equation of flow in a porous medium, a single conservation law, equations of Hamilton-Jacobi type, the Carleman model of two particle collinear scattering, the Stefan problem, nonlinear diffusion, variational and quasi-variational inequalities of evolution and many others already known or yet to be discovered. This framework is the core subject of Barbu's book.

To be more precise requires some notation. Let X be a Banach space with norm denoted by $\| \ \|$, and let $A\colon D(A) \subseteq X \to X$. Everywhere below $f\colon [0, T] \to X$ is a strongly (or Bochner) integrable function. Choosing $u_0 \in X$ we consider the Cauchy problem

$$CP(A, f, u_0)\colon \begin{cases} du/dt = A(u) + f(t), \\ u(0) = u_0, \end{cases}$$

defined by A, f and u_0. Here "$f(t)$" corresponds to adding a known source or forcing term to (EE); this is a useful generality, both because of the interpretation of f as a source term in models and because adequate knowledge of the way the solution of $CP(A, f, u_0)$ depends on f can be used to study perturbation problems. (Indeed, the "variation of parameters" formula

$$u(t) = (\exp tA)u_0 + \int_0^t (\exp(t - s)A) f(s)\, ds$$

for the solution of $CP(A, f, u_0)$ in the classical linear case finds its most striking applications in perturbation studies.)

The basic method of "solving" $CP(A, f, u_0)$ in our setting is via approximation by implicit difference schemes. For the purposes of this review, an ε-approximation to $CP(A, f, u_0)$ on $[0, T]$ by *an implicit difference scheme* is defined by a partition $0 = t_0 < t_1 < \cdots < t_n$ of $[0, t_n]$, a finite sequence $\{g_i\}_{i=1}^n \subseteq X$ and an $x_0 \in X$ satisfying

$$\begin{aligned}
&\text{(i)} \quad 0 \leq T - t_n \leq \varepsilon, \quad t_{i+1} - t_i \leq \varepsilon \quad \text{for } i = 1, \ldots, n,\\
&\text{(ii)} \quad \sum_{i=1}^{n} \int_{t_{i-1}}^{t_1} \|f(\tau) - g_i\| \, d\tau \leq \varepsilon,\\
&\text{(iii)} \quad \|x_0 - u_0\| \leq \varepsilon. \qquad (1)
\end{aligned}$$

A solution of the difference approximation defined by $\{0 = t_0 < t_1 < t_2 < \ldots < t_n\}$, $\{g_i\}_{i=1}^n$, and x_0 is a piecewise constant function v: $[0, t_n] \to X$ such that $v(\tau) = v(t_i)$ for $\tau \in (t_{i-1}, t_i]$ and

$$\begin{aligned}
&\text{(i)} \quad v(0) = x_0,\\
&\text{(ii)} \quad \frac{v(t_i) - v(t_{i-1})}{t_i - t_{i-1}} = A(v(t_i)) + g_i; \qquad i = 1, \ldots, n. \qquad (2)
\end{aligned}$$

One has:

THEOREM. *Assume there is an* $\omega \in \mathbf{R}$ *such that*

$$\|x - \hat{x} - \lambda(A(x) - A(\hat{x}))\| \geq (1 - \lambda\omega)\|x - \hat{x}\|$$

$$\textit{for } x, \hat{x} \in D(A), \textit{ and } \lambda > 0. \quad (3)$$

Let $u_0 \in D(A)$, $T > 0$ *and* v_k *be a solution of an* ε_k*-approximation to* $CP(A, f, u_0)$ *on* $[0, T]$ *by an implicit difference scheme for* $k = 1, 2, \ldots$. *If* $\lim_{k\to\infty} \varepsilon_k = 0$, *then there is exactly one continuous function* u: $[0, T] \to X$ *such that* $\lim_{k\to\infty}\|v_k(t) - u(t)\| = 0$ *uniformly on compact subsets of* $[0, T)$.

Assumption (3) is the only restriction on A, replacing the Lipschitz continuity and/or compactness one is accustomed to. Here are some simple examples for orientation: If A is Lipschitz continuous with constant L, then $\omega = L$ works; if $X = \mathbf{R}$ and A is nonincreasing, then $\omega = 0$ works; if X is a Hilbert space and A is linear, perhaps unbounded, selfadjoint and $A \leq 0$, or A is skew-adjoint, then $\omega = 0$ works. If (3) holds one says $A - \omega I$ is dissipative or, equivalently, $-A + \omega I$ is accretive. Some differential operators and spaces in which they have dissipative realizations are: (a) $A(u) = \Delta u$ in L^p, $1 \leq p \leq \infty$; (b) $A(u) = \Delta(u^\alpha)$, $\alpha > 0$, in L^1 and H^{-1}; (c) $A(u) = \sum_{i=1}^n (g_i(u))_{x_i}$ in L^1; (d) $A(u) = (\Delta u)^\alpha$, $\alpha > 0$, in L^∞; (e) $A(u) = \sum_{i=1}^n (\partial/\partial x_i)(|\partial u/\partial x_i|^{q-1}\partial u/\partial x_i)$ for $q \geq 1$ in L^p, $1 \leq p \leq \infty$; (f) $A(u) = g(\text{grad } u)$ in L^∞.

Assuming that the hypotheses of the above theorem are satisfied, we denote the limiting function u whose existence is asserted by the theorem by $K(A, f, u_0)$. Now one simply defines $u = K(A, f, u_0)$ to be the "solution" of $CP(A, f, u_0)$ on $[0, T]$ if $K(A, f, u_0)$ "exists" (i.e., the hypotheses of the theorem hold: in particular, (3) holds and there are solutions v_k of ε_k-approximate schemes with $\varepsilon_k \to 0$). The basic theory continues by discussing the following questions: When does $K(A, f, u_0)$ exist? What is the relationship between the notion of solution "$K(A, f, u_0)$" of $CP(A, f, u_0)$ and more classical notions which require the existence of derivatives? How does $K(A, f, u_0)$ vary as A, f, u_0 vary? Given a candidate function u, how can we tell if $u = K(A, f, u_0)$ or not? The answers to these questions constitute the fundamentals of the subject. In treating them, it is a handicap to restrict one's attention to functions. It turns out to be natural to let A be a "multifunction", i.e., to let A map X to the subsets of X. We now write Ax rather than $A(x)$, we agree that Ax is defined for every $x \in X$ and set $D(A) = \{x \in X: Ax \neq \emptyset\}$. Sums, scalar multiples and inverses of multifunctions are defined in the obvious way and functions are regarded as special cases of multifunctions. With slight modifications, the above theorem is sensible and true for multifunctions A. This generality turns out to be very useful in applications, costs nothing in terms of analytical difficulty and is essential in the formulation of some remarkable results as discussed below.

In contrast to the above implications that this subject may be regarded as basic ordinary differential equations for a class of applications in infinite dimensions, Barbu states in his preface that "The theory as developed below is a generalisation of the Hille-Yosida theory for one parameter semigroups of linear operators and is a collection of diversified results unified more or less loosely by their methods of approach." The book jacket adds: "The author's first aim is a survey of the basic methods and results on nonlinear semigroups theory, an approach with a strong unifying effect in the existence theory of nonlinear differential equations." The second statement is consistent with our view, the first is not. The Hille-Yosida theorem roughly corresponds to the special topic of linear equations in this subject just as classical ordinary differential equations has within it the special topic of linear equations. I feel it is a handicap to think of "nonlinear" as a generalization of "linear". But Barbu's assertion is historically accurate and this point of view is reflected in his presentation. It also led to the results described below.

If $R(I - \lambda A) \supset \overline{D(A)}$ for $\lambda > 0$ and the multifunction version of (3) holds with $\omega = 0$, then every difference approximation (2) with $x_0 \in \overline{D(A)}$ and $g_i = 0$, $i = 1, \ldots, n$, has a unique solution. Indeed $v(t_i)$ is determined from $v(t_{i-1})$ by solving $v(t_i) - (t_i - t_{i-1})Av(t_i) \ni v(t_{i-1})$. Thus $K(A, 0, u_0)$ exists for $u_0 \in \overline{D(A)}$. If one defines $S(t)u_0 = u(t)$ where $u = K(A, 0, u_0)$, then it is shown that $S(t)$: $\overline{D(A)} \to \overline{D(A)}$ for $t \geq 0$, $S(t)S(\tau) = S(t + \tau)$ for $t, \tau \geq 0$, $\|S(t)x - S(t)y\| \leq \|x - y\|$ for $t \geq 0$, $x, y \in \overline{D(A)}$ and $S(t)x \to S(0)x = x$ as $t \to 0+$ for $x \in \overline{D(A)}$. That is, $S(t)$ is a (strongly continuous) semigroup of contractions on $\overline{D(A)}$. Similarly one defines semigroups of contractions (all of which are assumed strongly continuous) on arbitrary sets $C \subset X$. If S is a semigroup of contractions on C, then its infinitesimal generator A_S is defined by $A_S x = \lim_{t \to 0+} t^{-1}(S(t)x - x)$, where $D(A_S)$ is the set on which the limit is defined.

Various famous theorems in linear functional analysis identify the infinitesimal generators of special classes of semigroups. For example, the infinitesimal generators of semigroups of unitary operators (i.e., each $S(t)$ is unitary) on a Hilbert space are precisely the skew-adjoint operators (Stone's theorem); the infinitesimal generators of semigroups of *linear* contractions are precisely the linear densely defined operators A satisfying (3) with $\omega = 0$ and $R(I - \lambda A) = X$ for $\lambda > 0$ (Hille-Yosida theorem); if X is also a Hilbert space H the Hille-Yosida conditions can be restated as: A is linear, densely defined and maximal with respect to extension among linear operators satisfying (3) with $\omega = 0$ (Phillips' theorem).

By contrast, we do not know any characterizations of the infinitesimal generators of semigroups of nonlinear contractions on a general Banach space X. We do know that there are semigroups of contractions with empty infinitesimal generators, and an outstanding open problem is whether or not $\{x \in X: \lim_{t\downarrow 0} t^{-1}\|S(t)x - x\| < \infty\}$ is nonempty whenever S is a semigroup of contractions on X. However, when X is a Hilbert space H there is a sharp theorem; this states that the set of infinitesimal generators of semigroups of contractions on closed convex subsets of H is precisely $\{A^0: A$ is a maximal dissipative multifunction in $H\}$. A^0 is the function which assigns to $x \in D(A)$ the element of least norm in Ax. Moreover, the associated correspondence between maximal dissipative multifunctions and semigroups is biunique. Here a "dissipative" multifunction is one satisfying the multifunction version of (3) with $\omega = 0$ and "maximal" refers to ordering by extension. Maximal dissipative multifunctions A in H are precisely the dissipative multifunctions satisfying $R(I - A) = H$ (Minty's theorem).

Simple examples show that infinitesimal generators are very delicate; the mildest perturbations destroy the property of being an infinitesimal generator. On the other hand maximal dissipative multifunctions are quite stable. For this and other reasons, even in Hilbert spaces one regards the multifunction giving rise to a semigroup as its "generator" (rather than the infinitesimal generator).

(The biunique correspondence mentioned above was suggested to me by R. T. Rockafellar after initial work of Y. Kōmura, T. Kato, J. R. Dorroh, A. Pazy and myself, and was proved by Pazy and myself. Y. Kōmura, who initiated this theory, established its most difficult link. He showed that the infinitesimal generator of a semigroup of contractions on closed convex subset C of H is densely defined in C. As mentioned before this is false in general spaces. The characterization of spaces in which it is true remains open. This theoretically fascinating problem is not of great applied significance, however: One is seldom presented with the solutions (that is, S) of a problem and asked for the problem (i.e., A) in nonlinear analysis.)

Thus the subject has many facets. Interpolate anywhere between the claims of applied significance (problems involving the heat equation, the ...) and mathematical elegance (... the correspondence between maximal dissipative multifunctions and semigroups is biunique ...), and there is something interesting to be found.

My view of this subject has evolved since the original 1974 Rumanian edition of Barbu's book appeared, partly due to the influence of works of J. Yorke and J. Kaplan, T. Takahashi, and Y. Kobayashi, which are not reflected in this book. The one theorem I stated above is proved in [3] and is not contained in the book (which presents the original special case obtained by myself and T. Liggett), and Barbu uses "generator" in the sense that I used "infinitesimal generator". A few words, then, about the book. The five chapters contain many topics I did not mention above. The first chapter is devoted to reviewing "preliminaries" (duality mappings, renorming theorems, strict and uniform convexity, vector-valued distributions, Sobolev spaces, Hille-Yosida theorem, etc.), and the second to basic results about maximal monotone operators, convex functions, subdifferentials, and dissipative operators in Banach spaces. There is a lot of material in these two preparatory chapters and it occupies nearly 100 pages. Chapter III corresponds best to the first part of my review; and Chapter IV contains the generation theorem in Hilbert spaces discussed above and goes on to discuss other topics in Hilbert spaces including regularity results for $u = K(-\partial\varphi, f, u_0)$ when φ is a convex functional, variational evolution problems, some nonlinear Volterra equations and other topics. Chapter V is devoted to a collection of topics unified by the appearance of second time derivatives.

This is a work of impressive scope. Barbu has collected a large body of recent research in his book and he has done a very considerable task in organizing it in a coherent way. However, I feel the book has shortcomings from the point of view of most readers. The prospect of plowing through 100 pages of preparations before hitting the main topic is discouraging, even if the material is of independent interest. The rich collection of examples is neither motivated nor put in perspective. Moreover, the presentation of the examples assumes an expertise in partial differential equations which will exclude many potential readers. The abstract developments can be followed independently of the examples, but this is a rather sterile approach to the subject.

In view of these considerations, my advice to the potential reader of Barbu's book would be to first look over [2] and then [4]. A skimming of [3] might also be worthwhile. These papers all appeared after the book and should facilitate reading it as well as provide some basic information and perspective without 100 pages of preliminaries. A principle omission of Barbu's book is that there is no treatment of the continuity of $K(A, f, u_0)$ with respect to A. A recent source for this is [5], while other references are given in [2].

The only other book available to most readers on this subject is Brezis' excellent monograph [1] concerning the Hilbert space theory. (I. Miyadera has published a volume in Japanese.)

Finally, there have been a variety of other developments in this area which are too recent to appear in Barbu's extensive bibliography and which I am not able to describe here. Recent work of J. B. Baillon, Ph. Benilan, H. Brezis, R. Bruck, L. C. Evans, A. Pazy and M. Pierre, among others, is relevant.

I wish to thank my friends for their comments on this review.

References

1. H. Brezis, *Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert*, Math. Studies no. 5, North-Holland, Amsterdam, 1973.

2. M. G. Crandall, *An introduction to evolution governed by accretive operators*, Dynamical Systems. Vol. 1: An International Symposium, Academic Press, New York, 1976, pp. 131–165.

3. M. G. Crandall and L. C. Evans, *On the relation of the operator $\partial/\partial s + \partial/\partial\tau$ to evolution governed by accretive operators*, Israel J. Math. **21** (1975), 261–278.

4. Y. Kobayashi, *Difference approximation of Cauchy problems for quasi-dissipative operators and generation of nonlinear semigroups*, J. Math. Soc. Japan **27** (1975), 640–665.

5. I. Miyadera and Y. Kobayashi, *Convergence and approximation of nonlinear semigroups*, Proc. Japan-France Seminar on Functional Analysis and Numerical Analysis, Tokyo, 1976 (to appear).

MICHAEL G. CRANDALL