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ON THE THEORY OF Π_3^1 SETS OF REALS

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1. An ordinal basis theorem. Assuming that $\forall x \in \omega^\omega$ ($x^\#$ exists), let u_α be the αth uniform indiscernible (see [3] or [2]). A canonical coding system for ordinals $< u_\omega$ can be defined by letting $WO_\omega = \{w \in \omega^\omega: w = \langle n, x^\# \rangle$, for some $n \in \omega, x \in \omega^\omega\}$ and for $w = \langle n, x^\# \rangle \in WO_\omega$, $|w| = \tau_n^{L[x]}(u_1, \ldots, u_{k_n})$, where τ_n is the nth term in a recursive enumeration of all terms in the language of $ZF + V = L[\dot{x}]$, $\dot{x}$ a constant, taking always ordinal values. Call a relation $P(\xi, x)$, where ξ varies over u_ω and x over ω^ω, Π_k^1 if $P^*(w, x) \Leftrightarrow w \in WO_\omega \wedge P(|w|, x)$ is Π_k^1. An ordinal $\xi < u_\omega$ is called Δ_k^1 if it has a Δ_k^1 notation i.e. $\exists w \in WO_\omega$ ($w \in \Delta_k^1 \wedge |w| = \xi$).

THEOREM 1 ($ZF + DC +$ DETERMINACY (Δ_2^1)). *Every nonempty Π_3^1 subset of u_ω contains a Δ_3^1* ordinal.

COROLLARY 2 ($ZF + DC +$ DETERMINACY (Δ_2^1)). *Π_3^1 is closed under quantification over ordinals $< u_\omega$ i.e. if $P(\xi, x)$ is Π_3^1 so are $\exists \xi P(\xi, x)$, $\forall \xi P(\xi, x)$.*

COROLLARY 3 ($ZF + DC + AD$). *The class of Π_3^1 sets of reals is closed under $< \delta_3^1$ intersections and unions.*

Martin [3] has proved the corresponding result for Δ_3^1.

2. A Kleene theory for Π_3^1. Kleene has characterized the Π_1^1 relations as those which are inductive (see [7]) on the structure $\langle \omega, < \rangle = \mathcal{Q}_1$. Let $j_m: u_\omega \longrightarrow u_\omega$, $m \geq 1$, be defined by letting

$$j_m(u_i) = \begin{cases} u_i, & \text{if } i < m, \\ u_{i+1}, & \text{if } i \geq m, \end{cases}$$

and then

$$j_m(\tau_n^{L[x]}(u_1, \cdots u_{k_n})) = \tau_n^{L[x]}(j_m(u_1) \cdots j_m(u_{k_n})).$$

Let R be the relation on u_ω coding these embeddings, i.e.

$$R = \{(m, \alpha, \beta): m \in \omega \wedge \alpha, \beta < u_\omega \wedge j_m(\alpha) = \beta\}.$$

Put $\mathcal{Q}_3 = \langle u_\omega, <, R \rangle$.

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THEOREM 4.($ZF + DC +$ DETERMINACY (Δ^1_2)). *A set of reals is Π^1_3 iff it is absolutely inductive on the structure $\mathcal{Q}_3$.*

In the second part of the above characterization a relation on reals is viewed as a second order relation on u_ω and absolutely inductive means that only parameters from ω are allowed in the definitions (see [7]).

It should be mentioned here that $\mathcal{Q}_3$ is up to absolute hyperelementary equivalence the same as $\langle u_\omega, <, T^2 \rangle$, where T^2 is the tree (on $\omega \times u_\omega$) coming from the Martin and Solovay [4] analysis of Π^1_2 sets (see [3] for the definition of T^2).

One also obtains the analog for Π^1_3 of the Souslin-Kleene representation of Π^1_1 sets in terms of well-founded trees.

THEOREM 5 ($ZF + DC +$ DETERMINACY (Δ^1_2)). *A set of reals P is Π^1_3 iff there is a tree T on $\omega \times u_\omega$ which is recursive in the structure $\mathcal{Q}_3$ and $P(x) \Leftrightarrow T(x)$ is well founded.*

For the notation see [2]. The fact that every Π^1_3 set can be so represented is a well-known result of Martin and Solovay [4], the converse being new here.

Let $\mathcal{Q}_3^- = \langle u_\omega, <, \{u_n\}_{n<\omega} \rangle$. Then we also have the context of full AD, in which case $u_n = \aleph_n$, $\forall n \leqslant \omega$.

THEOREM 6 ($ZF + DC + AD$). *A set of reals is Π^1_3 iff it is Π^1_1 on the structure $\mathcal{Q}_3^-$.*

3. Explaining the Q-theory. The results in §2 provide a nice explanation for the Q-theory (see [5], [1]) at level 3, which accounts for the structural differences between Π^1_3 and Π^1_1 sets. For example, a real is Δ^1_3 iff it is absolutely hyperelementary on $\mathcal{Q}_3$ while it is in Q_3 iff it is hyperelementary (i.e. parameters $< u_\omega$ are allowed) on $\mathcal{Q}_3$. Also if y_0 is the first nontrivial Π^1_3 singleton then y_0 is hyperelementary-in-$\mathcal{Q}_3$ equivalent to the complete inductive-in-$\mathcal{Q}_3$ subset of u_ω.

4. Higher level analogs of L. Assuming Projective Determinacy (PD), let T^3 be the tree (on $\omega \times \delta^1_3$) associated with an arbitrary Π^1_3-scale on a complete Π^1_3 set (see [6] and [2]). Let also C_4 be the largest countable Σ^1_4 set. The next result proves a conjecture of Moschovakis and shows that $L[T^3]$ is a correct higher level analog of L for level 4.

THEOREM 7 ($ZF + DC +$ DETERMINACY ($L[\omega^\omega] \cap$ power (ω^ω))). *For any T^3 as above, $L[T^3] \cap \omega^\omega = C_4$. In particular $L[T^3] \cap \omega^\omega$ is independent of the tree T^3.*

Open problem. Is $L[T^3]$ independent of T^3?

Further applications of the methods developed here to the theory of Π^1_3 sets as well as details and proofs of the results announced here will appear elsewhere.

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