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AVERAGE GAUSSIAN CURVATURE OF LEAVES OF FOLIATIONS

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Let $\mathcal{F}$ be a smooth transversely-oriented foliation of a compact, connected, oriented, Riemannian manifold W^{n+1} of constant sectional curvature $\equiv c$. Let $K_{\mathcal{F}}$: $W \longrightarrow \mathbf{R}$ via $K_{\mathcal{F}}(x)$ = the Gaussian curvature (defined below) of the leaf l^n through x at x. For $n = 2$ this is classical Gaussian curvature. Let vol be the canonical volume on W, and define $\overline{K}_{\mathcal{F}}$ by Volume $(W) \cdot \overline{K}_{\mathcal{F}} = \int_W K_{\mathcal{F}} \text{vol}$.

THEOREM 1.

$$\overline{K}_{\mathcal{F}} = \begin{cases} 2^n c^{n/2} \Big/ \binom{n}{n/2}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

THEOREM 2. *Let $n + 1 = 3$ and suppose $\mathcal{F}$, W, c are as above except that ∂W is nonempty and is a union of leaves of $\mathcal{F}$. Then*

$$\int_W K_{\mathcal{F}} \text{vol} = 2c \operatorname{Volume}(W) + \int_{\partial W} H \operatorname{vol}'$$

where H: $\partial W \longrightarrow \mathbf{R}$ is the mean curvature (computed with respect to the transverse orientation), and vol$'$ *is the canonical volume on ∂W.*

THEOREM 3. *Suppose $n + 1 = 3$. Let $\mathcal{F}$ and W be as in the original hypotheses with $\partial W = \varnothing$ but assume the sectional curvatures of W lie between c_1 and c_2. Then we have $2c_1 \leqslant \overline{K}_{\mathcal{F}} \leqslant 2c_2$.*

DEFINITION OF GAUSSIAN CURVATURE. We define, for a Riemannian manifold $l = l^n$, the function K: $l \longrightarrow \mathbf{R}$ in two cases (which overlap):

Case (i). n is even. In this case a local orthonormal frame on l gives rise to a matrix of curvature 2-forms, $\Omega = (\Omega^i_j)$ defined locally. The Pfaffians of the local Ω agree on overlaps and so define a global n-form Pf(Ω) on l. Letting ν denote the canonical volume form on l we set

$$K\nu = \frac{2^{n/2} \cdot (n/2)!}{n!} \operatorname{Pf}(\Omega)$$

(see [3, vol. V, pp. 417–420]).

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Case (ii). Assume l is a hypersurface of a flat Riemannian manifold W, and that l is transversely oriented by a field of unit normals ξ. Then at each point x of l let A_x: $T_x l \longrightarrow T_x l$ be defined by $A_x v = -\nabla_v \xi$. Then we define $K(x) = \det(A_x)$. (See [3, vol. IV, p. 96].)

REMARKS. In the overlap of Cases (i) and (ii), viz. when l is an even-dimensional hypersurface of a flat manifold, the two definitions of K agree. If n is even then K is intrinsic to the geometry of l; if $n \geqslant 3$ is odd then K is intrinsic up to a global choice of sign [3, vol. IV, p. 96].

SKETCH OF PROOF OF THEOREM 1. We consider two cases: n odd and n even.

(i) The case n is odd:

Here $\chi(W) = 0$ and hence by Chern-Gauss-Bonnet [3, vol. V, p. 442] the constant curvature $c = 0$, i.e. W is flat. Without loss of generality we may assume, by taking a finite covering, that W is in fact a flat torus [1, p. 212].

Let $T_p \approx \mathbf{R}^{n+1}$ denote the tangent space to W at some point $p \in W$. A choice of unit normal vector field ξ to the foliation F determines (by parallel translation in W) a Gauss map $g: W \longrightarrow T_p$ whose image lies of course in the unit sphere $S^n \subset T_p$. Think of dg as a map $dg: W \longrightarrow \mathrm{End}(TW)$ via $x \mapsto dg_x$.

Let $\sigma_i(E_x)$ denote the ith elementary symmetric function of the eigenvalues of E_x, where E_x is any endomorphism E_x: $T_x \longrightarrow T_x$.

LEMMA. *$K_F(x) = \sigma_n(-dg_x)$, for all $x \in W$.*

The proof is not difficult.

Now for each $t \in \mathbf{R}$ consider h_t: $W \longrightarrow W$ defined by $h_t(x) = \exp(tg(x))$, or in other words $h_t(x) = x + tg(x)$ (by slight abuse of notation). A computation shows that

$$\int_W Jh_t \mathrm{vol} = \int_W \det(I + tdg)\mathrm{vol} \quad \text{or}$$

$$(*) \qquad \mathrm{Volume}(W) = \mathrm{Volume}(W) \cdot [1 + \bar{\sigma}_1(dg)t + \cdots + \bar{\sigma}_n(dg)t^n]$$

where $\bar{\sigma}_i(dg)$ denotes the average over $x \in W$ of $\sigma_i(dg_x)$, and J denotes the Jacobian.

Since both sides of (*) are polynomials in t it follows that $\bar{\sigma}_i(dg) = 0$, $i = 1, \ldots, n$.

COROLLARY. *In the above case we have $\bar{\sigma}_i(dg) = 0$ for $i = 1, \ldots, n$. In particular $\sigma_2(dg)$ is a multiple of the leaf scalar curvature; hence the average leaf scalar curvature is 0 whenever W is flat.*

SKETCH OF PROOF OF THEOREM 1 (CONTINUED).

(ii) The case n is even:

The proof depends on the construction of certain globally defined n-forms. Let $\{\theta^1, \ldots, \theta^n, \theta^{n+1}\}$ be a local adapted orthonormal coframe field (with

θ^{n+1} orthogonal to the leaves of F) and let $\{\omega_j^i\}$ be the associated Riemannian connection forms. Put

$$\phi_r = \sum_{\sigma \in S_n} (-1)^\sigma \omega_{n+1}^{\sigma(1)} \wedge \cdots \wedge \omega_{n+1}^{\sigma(2r-1)} \wedge \theta^{\sigma(2r)} \wedge \cdots \wedge \theta^{\sigma(n)}$$

for $1 \leqslant r \leqslant n/2$, where S_n denotes the symmetric group on $\{1, \ldots, n\}$ and $(-1)^\sigma$ is the sign of the permutation σ.

LEMMA. *The n-forms ϕ_r do not depend on the choice of orthonormal coframe $\{\theta^i\}$ and hence are globally defined on W.*

The proof is an unpleasant calculation.

LEMMA. *For each n there exist constants b_r, $1 \leqslant r \leqslant n/2$ such that if we set*

$$\Phi = \sum_{r=1}^{n/2} b_r \phi_r \qquad \textit{then}$$

(**)

$$d\Phi = (K_F - a_n c^{n/2})\mathrm{vol} \quad \textit{where } a_n = 2^n \Big/ \binom{n}{n/2}.$$

The proof is an even more unpleasant calculation.

Integrating (**) over W readily yields $\bar{K}_F = 2^n c^{n/2} / \binom{n}{n/2}$ as desired.

REMARKS. By taking double covers we may prove Theorem 1 even if W is allowed to be nonorientable. If n is even then we may similarly drop the assumption that F is transversely orientable. If n is odd, however, transverse orientability is required in order that K_F be defined.

Theorem 1 has been generalized in various ways in the recent paper of Rosenberg, Brito and Langevin [2]. Theorems 2 and 3 are proved using methods similar to Theorem 1.

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