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Problems and theorems in analysis, by G. Pólya and G. Szegö, Die Grundlehren der math. Wissenschaften, Springer-Verlag, Berlin and New York; Vol. I, 1972, xix + 389 pp., Vol. II, 1976, xi + 391 pp., $45.10.

Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis* was published first in 1925 as volumes 19 and 20 of the "yellow-peril" series. See Tamarkin [**1**] for a review. The inexpensive reprint in 1945 (Dover Publications) by authority of the U. S. Alien Property Custodian made the work widely known in N. America. The four Springer (German) editions through the latest (1970, 1971) are unchanged from the original except for the correction of minor errors.

The present volumes are a revised and enlarged translation of the 4th edition, vol. I translated by Dorothee Aeppli and vol. II by Claude E. Billigheimer.

The work is one of the real classics of this century; it has had much influence on teaching, on research in several branches of hard analysis, particularly complex function theory, and it has been an essential indispensable source book for those seriously interested in mathematical problems. One can think of few books written more than a half century ago that would really be worth translating today. This one certainly was; of course some parts are a bit faded and dated, but much is fresh and exciting and will be consulted for years to come. The translators (whose work is first-rate), authors, and publisher deserve our praise for making Pólya-Szegö available in English to the ever widening set of mathematicians and students who no longer read German.

These volumes contain many extraordinary problems and sequences of problems, mostly from some time past, well worth attention today and tomorrow. Before embarking on my reviewer's responsibility of evaluation and criticism, I want to emphasize, regardless of anything I say below, my personal enormous respect for the mathematics of Pólya-Szegö. This work was written in the early twenties by two young mathematicians of outstanding talent, taste, breadth, perception, perseverence, and pedagogical skill. It broke new ground in the teaching of mathematics and how to do mathematical research.

Structure. I shall assume the reader is not yet familiar with Pólya-Szegö. I cannot explain in better words than the authors' what their object was; so I quote several paragraphs from the Preface (to the first edition):

> The chief aim of this book, which we trust is not unrealistic, is to accustom advanced students of mathematics, through systematically arranged problems in some important fields of analysis, to the ways and means of independent thought and research. It is intended to serve the need for individual active study on the part of both the student and the teacher. The book may be used by the student to extend his own reading or lecture material, or he may work quite independently through selected portions of the book in detail. The instructor may use it as an aid in organizing tutorials or seminars.
>
> This book is no mere collection of problems. Its most important feature is the systematic arrangement of the material which aims to stimulate the reader to independent work and to suggest to him useful lines of thought. We have devoted more time, care and detailed effort to devising the most effective presentation of the material than might be apparent to the uninitiated at first glance.
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> The imparting of factual knowledge is for us a secondary consideration. Above all we aim to promote in the reader a correct attitude, a certain discipline of thought, which would appear to be of even more essential importance in mathematics than in other scientific disciplines.
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> One should try to understand everything: isolated facts by collating them with related facts, the newly discovered through its connection with the already assimilated, the unfamiliar by analogy with the accustomed, special results through generalization, general results by means of suitable specialization, complex situations by dissecting them into their constituent parts, and details by comprehending them within a total picture.
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> An idea which can be used only once is a trick. If one can use it more than once it becomes a method. In mathematical induction the result to be obtained and the means available for its proof are proportional, they stand in the ratio of $n + 1$ to n. Hence, strengthening the statement to be proved may also be advantageous, for we strengthen at the same time the means available for its proof. It is also found in other circumstances that the more general statement may be easier to prove then the more particular; in such cases the most important achievement consists precisely in setting up the more general statement, in extracting the essential, in realizing the complete picture.
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> . . . .
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> However, one must not forget that there are two kinds of generalization, one facile and one valuable. One is generalization by dilution, the other is generalization by concentration. Dilution means boiling the meat in a large quantity of water into a thin soup; concentration means condensing a large amount of nutritive material into an essence. The unification of concepts which in the usual view appear to lie far removed from each other is concentration. Thus, for example, group theory has concentrated ideas which formerly were found scattered in algebra, number theory, geometry and analysis and which appeared to be very different. Examples of generalizations by dilution would be still easier to quote, but this would be at the risk of offending sensibilities.

In this new edition, the authors have added about 160 new problems (making a total of about 1820 problems) and have modified the statements or solutions of about a dozen old problems.

The collection is organized into nine parts, each divided into chapters and sections. Volume I contains parts 1–3 followed by the solutions; volume II the remaining problems and solutions. Each volume has its own author and subject indices. Volume II contains an appendix of 9 additional problems to part I (of volume I) and their solutions, a list of the numbers of all new problems in both volumes, a list of topics that are not evident in the arrangement of the books, but are represented by connected series of problems, and finally a brief Errata for volume I.

Content. I shall list with some detail the subject matter, however, a list does not really tell everything. There are threads that run through many topics and pop up in unexpected places. For example there is a thread of combinatorial methods and generating functions; there is a thread of inequalities; there is a thread of geometry, and so on. One of the joys in browsing through Pólya-Szegö is the cross-referencing. Start anywhere and work on a few problems; read the authors' solutions. You are invariably referred to something related on other pages, and from there you are sent elsewhere, etc. The authors' painstaking care in showing interconnections is quite remarkable.

As I go over the contents, I shall choose a few samples that indicate both the quality and flavor of the material.

Part I, *Infinite series and infinite sequences*, includes generating functions, transformations of series, partition identities, and much on convergence of real series and sequences. Here is one attractive example (173): Let $0 < x_0 \leqslant 1$ and set $x_{n+1} = \sin x_n$. Then $\sqrt{n}\, x_n \to \sqrt{3}$. The solution by E. Jacobsthal could be improved to give the rate of convergence.

Part 2, *Integration*, starts with upper and lower sums and soon goes into the rate of convergence of Riemann sums. Already (15) we are asked to prove

$$e^{n/4} n^{-(n+1)/2} (1^1 2^2 3^3 \cdots n^n)^{1/n} \to 1.$$

A bit later (61):

$$\iint_{0 \leqslant x \leqslant y \leqslant \pi} \log |\sin(x-y)|\, dx\, dy = -\tfrac{1}{2}\pi^2 \ln 2.$$

The section on inequalities would be ideal for a course following the Pólya-Szegö method of instruction. Series problems more or less alternate with the analogous integral problems, and an ideal number of by-ways are explored. For example (94.2): Show that the surface area E of an ellipsoid with semiaxes a, b, c satisfies

$$\tfrac{4}{3}\pi(bc + ca + ab) \leqslant E \leqslant \tfrac{4}{3}\pi(a^2 + b^2 + c^2).$$

There is also material on bounded variation, convexity, Dirichlet kernels, $\Gamma(x)$, the Weierstrass approximation theorem, equidistribution, and asymptotics.

Parts 3 and 4, *Functions of one complex variable*, form the central core of the work. Together they contain about 560 problems, and there is a substantial overflow of function theory material into the next two parts. The emphasis is on geometric function theory. Two examples from part 3: First (299, 300) let $f_1(z), \ldots, f_n(z)$ be regular on $\mathbf{D}$ and set $\phi(z) = \sum |f_i(z)|$. Prove

$\phi(z)$ takes its maximum on $\partial \mathbf{D}$ and only on $\partial \mathbf{D}$ unless all the f_i are constant. Next (301), given $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbf{R}^3$, set $\phi(\mathbf{x}) = \Pi|\mathbf{x} - \mathbf{a}_i|$ for $\mathbf{x}$ in a domain $\mathbf{D} \subset \mathbf{R}^3$. Then $\phi(\mathbf{x})$ takes its maximum on $\partial \mathbf{D}$.

Part 5, *The location of zeros*, includes Descartes' rule of signs and many related topics and applications. For instance (76), let $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \beta_1 < \cdots < \beta_n$. Prove $\det[\exp(\alpha_i \beta_j)] > 0$.

Part 6, *Polynomials and trigonometric polynomials*, has the polynomials of Tchebychev, Legendre, Laguerre, Hermite, etc., with much material on zeros, signs, orthogonality, and extrema. Example (46): Let $f(x)$ be a real polynomial of degree n such that $f(x) \geqslant 0$ for $-1 \leqslant x \leqslant 1$. Then $f(x) = p(x)^2 + (1 - x^2)q(x)^2$, where $\deg p = n$ and $\deg q = n - 1$. Again (103) let $f(x)$ be real of degree n such that $\int_{-1}^{1} f^2\, dx = 1$. Then $|f(x)| \leqslant (n + 1)/\sqrt{2}$ for $|x| \leqslant 1$.

Part 7, *Determinants and quadratic forms*, deals mostly in special determinants and finite and infinite matrices that arise in analysis, special functions, and inequalities. There are interesting examples on positive definiteness.

Part 8, *Number theory*, with about 320 problems is the second largest topic in the books. There is material on number theoretic functions, Dirichlet series, polynomials and power series with integer coefficients, lattice points, and algebraic integers. Three problems of I. Schur (121, 123, 124) are nice: let $a_1, \ldots, a_n$ be distinct integers and $f(x) = \Pi(x - a_i)$. Then $f(x) - 1$, $f(x)^2 + 1$, and $f(x)^4 + 1$ are $\mathbf{Z}$-irreducible. (The proof of the last by A. and R. Brauer is not easy.) One section centers on a deep theorem of Eisenstein (140–154): If $f(z) = \Sigma_1^\infty a_n z^n$ has rational coefficients and is an algebraic function of z, then for some integer $N \geqslant 1$, $f(Nz)$ has integer coefficients.

Part 9, *Some geometric problems*, with only 34 problems is all too short.

This brief list of topics hardly begins to tell how much material there really is between the covers of these two volumes. The method of presentation and the brevity of most solutions creates an unusually high information density.

The average solution is just a few lines stating the key ideas. A few solutions give merely a reference to the source. Occasionally a detailed solution is given, and rarely there are two or three alternative solutions.

New material. There has been no attempt to update the material of the original edition. The new problems and solutions were meant to supplement in kind what was already there. Their distribution is interesting: 61 in part 1, mostly combinatorial, 50 in part 8 (number theory), the rest scattered, but only 19 in the two big parts on complex functions.

I find on the average that the new material is easier than the old: more hints, more routine exercises. Perhaps the authors felt a need to soften their work a bit for its modern audience.

Actually, the work as a whole and the new material in particular, must be evaluated within the context of Professor Pólya's creed on problem solving as presented in his books [2], [3], [4] and in many articles and lectures. The student who intends to use Pólya-Szegö seriously will be wise to browse through Pólya's books first. They are meant for a less sophisticated audience, but that makes them easier to read than the volumes under review.

Present value of the work. Although the separation is not always clear, there are two kinds of problems in Pólya-Szegö. First there are sequences of problems designed for the reader partially to discover for himself the basic ideas and results of a topic. In many cases, these have lost value because of changes in the topic over the last 50 years. For example, one would not ask a student to commit a large amount of time to the real analysis in parts 1 and 2 of Pólya-Szegö because of their omission of the Lebesgue integral and anything at all on functional analysis. A more concrete example is the sequence of exercises (volume II, pp. 146–147) on the g.c.d. of algebraic integers. They are the wrong questions in the wrong context.

Second are simply outstanding problems (or short chains of problems), maybe not important results in themselves. To my thinking, the following are the ingredients of a good mathematical problem: (1) the statement should be fairly short, involving not much structure and certainly not a batch of definitions, (2) the conclusion should be pleasing, (3) the problem should be challenging, not solvable by a straight-forward application of standard methods or by an ugly calculation, (4) most important, the solution must require at least one ingenious idea.

Pólya-Szegö is rich in problems of this type. In the past, more of the leading mathematicians proposed and solved problems than today, and there were problem departments in many journals. Pólya and Szegö must have combed all of the large problem literature from about 1850 to 1925 for their material, and their collection of the best in analysis is a heritage of lasting value.

Obviously, this is a rich source for problem seminars. On the value of teaching problem solving, I refer again to Pólya's books [2], [3], [4]. May I add my own opinion that doing mathematical research requires far more technique (proving theorems and constructing examples) than strategy (generalizations and conjectures). An analogy is the maxim (attributed to R. Fine) that chess is 10% strategy and 90% tactics. Problems solving is possibly the most efficient way to acquire and sharpen technical skills in mathematics.

There are other collections of problems. I give a very incomplete list [5]–[16] below, with brief comments.

I am obliged to mention two small points that detract from the usefulness of the new edition. As noted already, a few of the solutions consist only of a reference to the literature. Now many college and university libraries do not own (often obscure) journals of 50 and more years ago, so these solutions are not readily available to all. It would have been a service to have included sketches of the missing solutions–that would not have lengthened the work appreciably.

My other point is that the subject indices are needlessly brief. They should have been longer by about a factor of three. One instance of their inadequacy will suffice: I noticed the Riemann surface of $\ln z$ mentioned without explanation in problem 337 of part 3 and wanted to locate its source. Neither index has Riemann surface, surface Riemann, logarithm, nor any other clue. Incidentally, the reference to Cesàro on p. 376 of volume 2 is to E. Cesàro, *Elementares Lehrbuch der algebraischen Analysis* . . . , Teubner, 1904. (This was omitted from the translation.)

The work is unashamedly dated. With few exceptions, all of its material comes from before 1925. We can judge its vintage by a brief look at the author indices (combined). Let's start on the C's: Cantor, Carathéodory, Carleman, Carlson, Catalan, Cauchy, Cayley, Cesàro, ... Or the L's: Lacour, Lagrange, Laguerre, Laisant, Lambert, Landau, Laplace, Lasker, Laurent, Lebesgue, Legendre, ... Omission is also information: Carlitz, Erdös, Moser, etc.

Some comments on solutions. A few of the solutions were improved in this edition. For instance, (part 1, 99) given $a_m + a_n - 1 < a_{m+n} < a_m + a_n + 1$, prove $\{a_n/n\}$ converges, etc. A second solution was added that is really elegant compared with the first one. I suspect that many more of the problems have alternative better solutions, because of the inventiveness of problem solvers and because many topics have been generalized and are seen now in a broader context (without diluting the soup). Some problems are today rather easier than they probably were intended originally.

For instance, (part 7, 35): Given positive definite quadratic forms $\Sigma a_{ij}x_ix_j$ and $\Sigma b_{ij}x_ix_j$, to prove $\Sigma a_{ij}b_{ij}x_ix_j$ is positive definite. Today the (conceptual rather than computational) proof is to note that $[a_{ij}b_{ij}]$ is a principal submatrix of the positive definite tensor product $[a_{ij}] \otimes [b_{kl}]$.

Pólya's famous theorem (part 8, 239) on not seeing the forest for the trees is usually proved now by applying Minkowski's theorem to a $2r \times 2L$ rectangle centered at the origin. If it contains no further lattice point, then its area is less than 4, hence $4rL < 4$, $L < 1/r$.

Problem 3 of part 7 requests a proof of Cauchy's formula for an $n \times n$ determinant:

$$\det\left[\frac{1}{a_i + b_j}\right] = \prod_{i<j}(a_i - a_j)(b_i - b_j)\Big/\prod(a_i + b_j).$$

The given solution involves addition and subtraction of rows and columns to reduce to the case $n - 1$, etc. Not hard, but not very illuminating either because the formula seems an accidental by-product of computing. It would be nice to see without computing that the answer is correct in form, if not in all details. To this end, let $f(\mathbf{a}, \mathbf{b})$ be the desired determinant and $h(\mathbf{a}, \mathbf{b}) = \Pi(a_i + b_j)$. Then hf is a homogeneous polynomial of degree $n^2 - n$ in $(\mathbf{a}, \mathbf{b})$. It vanishes if $a_i = a_j$ or $b_i = b_j$, hence it is divisible by all $a_i - a_j$ and all $b_i - b_j$. Since $g(\mathbf{a}, \mathbf{b}) = \Pi(a_i - a_j)(b_i - b_j)$ has degree $n^2 - n$, it follows that $hf = c_n g$, where c_n is a constant.

Thus by talking and hand waving only, we have reduced the problem to its essence, the evaluation of c_n. This may be as hard as the original problem (it isn't), but that is not so important as our having made the formula plausible.

My remaining remarks concern part 9, *Geometric problems*. Good problems in geometry are hard to come by, particularly in differential geometry and out of the plane. This short section contains some gems, a few grinds best forgotten, and some solutions that seem unnatural today. I hope some detailed comments on one small batch of problems will indicate how much real meat the bones carry, and what challenge and pleasure an instructor of a problem seminar can have with almost any batch of Pólya-Szegö problems.

Problem 1.1 asks for the area S of the lateral surface of a circular cone of semiapex angle α between the apex and a plane that cuts the cone in an ellipse. To prove

$$S = \tfrac{1}{2}\pi(p + q)\sqrt{pq}\,\sin\alpha,$$

where p and q are distances from the apex to the ends of the major diameter of the ellipse. The authors' solution involves a nice idea, but their use of rectangular coordinates entails a nasty computation. The computation can be greatly reduced by using cylindrical coordinates: Let the cone be $r/A + z/C = 1$ and the plane $z/C = x/B = r(\cos\theta)/B$. Assume $0 < A < B$ and $C < 0$. The projection of the ellipse of intersection on the plane $z = 0$ is obtained by eliminating z: $r(A^{-1} + B^{-1}\cos\theta) = 1$. This is the polar equation of an ellipse. Now the ellipse with foci $(0, 0)$ and $(-c, 0)$ and semiaxes $a > b$, so $a^2 = b^2 + c^2$, is $r(a + c\cos\theta)b^{-2} = 1$, so $A = b^2/a$, $B = b^2/c$ and easily $b^2 = A^2B^2/(B^2 - A^2)$, $a^2 = A^2B^4(B^2 - A^2)^2$. Therefore

$$S = \frac{1}{\sin\alpha}\pi ab = \frac{\pi}{\sin\alpha}\,\frac{A^2B^3}{(B^2 - A^2)^{3/2}}\,.$$

This is essentially the solution. It is routine plane analytic geometry to derive $p = ab/(b - a)\sin\alpha$ and $q = ab/(b + a)\sin\alpha$, etc.

Problem 1.2 asks for the volume, surface area, and integrated mean curvature for the convex hull of two externally tangent spheres. Since no solution is given really, may I suggest first evaluating these quantities for an ice cream cone (convex hull of sphere and external point) and then using the well-known Steiner formulas for parallel bodies.

Problem 2 concerns a C'' function $f(x, y) \geqslant 0$ on $x^2 + y^2 \leqslant 1$, $f = 0$ on the boundary, f is not identically 0 inside. To prove $\pi|\operatorname{grad} f| \leqslant 3\int\int f$ everywhere is impossible. The solution is not given, so I offer the following. Use polar coordinates. Then $|\operatorname{grad} f|^2 = f_r^2 + r^{-2}f_\theta^2$. Set $V = \int\int f$. If $\pi|\operatorname{grad} f| \leqslant 3V$, then $\pi|f_r| \leqslant 3V$, hence

$$f(r, \theta) = |f(1, \theta) - f(r, \theta)| = \left|\int_r^1 f_r(t, \theta)\,dt\right| \leqslant \frac{3}{\pi}(1 - r)V,$$

$$V = \int\int rf\,dr\,d\theta \leqslant (3/\pi)V(2\pi)\int_0^1 r(1 - r)\,dr = V, \quad \text{etc.}$$

Problem 5 hardly needs the big guns of a Fourier expansion, since

$$\int_0^{2\pi} \rho(\theta)(\cos\theta, \sin\theta)\,d\theta = \int_0^{2\pi} \rho\mathbf{n}\,d\theta = \oint \mathbf{n}\,ds = \oint d\mathbf{t} = \mathbf{0}$$

proves the vanishing of the first moments. It follows that the zeroth through second moments of the function $\rho(\theta + \pi) - \rho(\theta)$ vanish, so a closed convex curve indeed has three point pairs with parallel tangents and equal radii of curvature.

Problems 6 and 7 and their solutions are extremely attractive. Problems 8–12 are essentially restatements of standard integral formulas in curve and surface theory. Written in *vector* form, they are pretty transparent.

Problem 13 asks for a proof that the spherical image of a closed smooth space curve meets every great circle, and the integral proof (C. Loewner) is the standard one we all know. But what a missed opportunity to throw in the beautiful result (due to Fenchel I think) that a closed curve on the unit sphere that meets every great circle has length at least 2π, and Milnor's extensions for knotted curves.

The next two problems form an instructive example of the value of choosing the optimal variables and frames–a key tool in modern differential geometry. Problem 15: "On a surface of revolution with continuous curvature there are always two different parallel circles with the same Gaussian curvature." Problem 14 is the main step in the solution: Given $f(x)$ continuous on $[a, b]$, $f(a) = f(b) = 0$, $f(x) > 0$ and $f \in C''$ on (a, b), $f'(x) \to \infty$ as $x \to a$, $f'(x) \to -\infty$ as $x \to b$. To prove that $F(x) = f''(x)/f(x)[1 + f'(x)^2]^2$ is not monotone in (a, b) unless $f(x) = [(x - a)(b - x)]^{1/2}$; then $F(x) = -4/(b - a)^2$. The result is due to Loewner. The solution of 14 involves some delicate partitioning of the interval and several applications of mean value theorems. The transition to 15 works via rotating $y = f(x)$ around the x-axis. After some tedious calculations with principal curvatures, it turns out that $F(x)$ is the Gaussian curvature.

I claim both problem 14 and its solution are unnatural. First, the curve $y = f(x)$ should be parameterized in terms of its arc length. Thus we consider $\mathbf{x} = \mathbf{x}(s) = (x(s), y(s))$, differentiable for $0 \leqslant s \leqslant L$, C'' for $0 < s < L$, and satisfying $\mathbf{x}(0) = (-a, 0)$, $\mathbf{x}(L) = (a, 0)$, $\mathbf{x}'(0) = (0, 1)$, $\mathbf{x}'(L) = (0, -1)$, $(x')^2 + (y')^2 = 1$. The surface of revolution is naturally parameterized by s and the rotation angle θ:

$$\mathbf{X}(s, \theta) = (x(s), y(s)\cos\theta, y(s)\sin\theta).$$

From

$$d\mathbf{X} = (x', y'\cos\theta, y'\sin\theta)ds + (0, -y\sin\theta, y\cos\theta)d\theta$$

we deduce that $\sigma_1 = ds$, $\sigma_2 = y\,d\theta$ is an orthonormal basis of one-forms on the surface. To find the Gaussian curvature, we require the unique one-form ω such that $d\sigma_1 = \omega \wedge \bar{\sigma}_2$ and $d\sigma_2 = -\omega \wedge \sigma_1$; then $d\omega + K\sigma_1 \wedge \sigma_2 = 0$. Easily, $\omega = y'\,d\theta$, $K = K(s) = -y''/y$. (This expression is more tractable than $F(x)$ above.) Since K is assumed continuous at 0 and L, we must have $y''(0+) = y''(L-) = 0$.

Now assume $K(s)$ is monotone, say decreasing. Since $y'(s) \geqslant -1$ and $y'(L) = -1$, it follows that $y''(s) \leqslant 0$ for values of s arbitrarily close to L, hence $K(s) \geqslant 0$ for all s. Therefore $y'' \leqslant 0$ on $(0, L)$ so y' is monotone decreasing also. If not strictly so, then there must be an interval $[c, L]$ where y' is constant. But this evidently forces $y'(s) = 1$, $z'(s) = 0$ on $[c, L]$, impossible. We conclude that $0 \leqslant s < t \leqslant L$ implies $y'(s) - y'(t) > 0$. Now consider

$$I = \int\int y(s)y(t)[K(s) - K(t)][y'(s) - y'(t)]\,ds\,dt,$$

taken over $[0, L] \times [0, L]$. (This device is parallel to Pólya's beautiful solutions to problems 6 and 7.) By direct computation $I = 0$. But the integrand is nonnegative, so we conclude that $K(s) = k^2$ is constant. It is routine to solve $y'' + k^2y = 0$ and eventually conclude that the original curve is a semicircle.

The solution of 17 refers to a measure on the space of planes in $\mathbf{R}^3$ that may be unfamiliar to some. It is easier to work with *directed* planes. Such a plane has a unique normal equation $\mathbf{n} \cdot \mathbf{x} = p$, where $\mathbf{n} \in S^2$. The measure is $d\mu = d\omega \wedge dp$, where $d\omega$ is the area element on S^2. Then $d\mu$ is invariant under the proper motion group. The solution incidentally is a gem.

Problem 17.1 asks for a proof of $A \leqslant \frac{1}{4}\sqrt{3}\,(abc)^{2/3}$, where a, b, c are the sides of a triangle. The solution works with the similarity class of triangles, parameterized by $x = b/a$ and $y = \cos\gamma$, expresses $A^6/(abc)^4$ in terms of x and y, and grinds away. If this method shed any light on the generalization to tetrahedra in 17.4, I could accept it, but it doesn't for me; anyhow I prefer a solution with some symmetry.

From $A = \frac{1}{2}ab\sin\gamma$ etc., we have

$$A^3 = \tfrac{1}{8}(abc)^2 \sin\alpha\,\sin\beta\,\sin\gamma.$$

This reduces the problem to proving

$$\sin\alpha\,\sin\beta\,\sin\gamma \leqslant \tfrac{3}{8}\sqrt{3}$$

for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha + \beta + \gamma = \pi$. This is easy, say by Lagrange multipliers. Equality obtains only for the equilateral triangle. Incidentally, an interesting related result is the inequality

$$\sin\alpha\,\sin\beta\,\sin\gamma \leqslant \left(3\sqrt{3}/2\pi\right)^3\alpha\beta\gamma$$

for the angles of a triangle.

One final observation: there is a little humor here and there in the work. I particularly recommend the solution of AI 191.2 on p. 382, volume II, for a laugh.

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11. J. G. Krzyz, *Problems in complex variable theory*, American Elsevier, New York, 1971, xix + 283 pp. [For supplementing a regular course. Mostly exercises, but some real problems.]

12. Ya. I. Rivkind, *Problems in mathematical analysis*, Noordhoff, Groningen, ca 1965, v + 98 pp. [Meant to supplement real variable courses; mostly routine on the hard side.]

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14. W. Sierpiński, 250 *problems in elementary number theory*, American Elsevier, New York, 1970, vii + 125 pp. [About half routine, half challenging problems, a few quite challenging.]

15. G. Szász et al., *Contests in higher mathematics, Hungary* 1949–1961, Akadémiai Kiadó, Budapest, 1968, 260 pp. [A collection of highest quality.]

16. A. M. Yaglom and I. M. Yaglom, *Challenging mathematical problems with elementary solutions*, Holden-Day, San Francisco, Calif.; Vol. I: *Combinatorial analysis and probability theory*, 1964, ix + 231 pp.; Vol. II: *Problems from various branches of mathematics*, 1967, xi + 214 pp. [A collection of outstanding problems at the U. S. university level.]

HARLEY FLANDERS

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Rekursive Funktionen in der Komputer Theorie, by Rózsa Péter, Akadémiai Kiadó, Budapest, Hungary, 1976, 190 pp., $12.00.

The Theory of Recursive Functions developed in its present form in the decades following 1930. Pioneered by the work of Turing, Post and Church, it has aimed at making precise and at studying the notions of algorithm and computation.

A (partial) function from the set of natural numbers into natural numbers is *recursive* if it can be represented by an expression formed from certain *base functions* and the operations of *substitution, primitive recursion*, and *minimization*. The base functions comprise the *successor function* ($S(x) = x + 1$), the *null function* ($N(x) = 0$), and *projection functions* ($U_i^n(x_1, \ldots, x_n) = x_i$, where $1 \leqslant i \leqslant n$). Primitive recursion is used to define a function $h(z, x_1, \ldots, x_n)$ from recursive functions $f(x_1, \ldots, x_n)$ and $g(z, y, x_1, \ldots, x_n)$ by the pair of equations

$$h(0, x_1, \ldots, x_n) = f(x_1, \ldots, x_n),$$
$$h(S(z), x_1, \ldots, x_n) = g(z, h(z, x_1, \ldots, x_n), x_1, \ldots, x_n).$$

The operation of minimization defines a (possibly partial) function $f(x_1, \ldots, x_n)$ from a total recursive function $g(y, x_1, \ldots, x_n)$ as the "smallest y such that $g(y, x_1, \ldots, x_n) = 0$," and is written

$$f(x_1, \ldots, x_n) = (\mu y)[g(y, x_1, \ldots, x_n) = 0].$$

Note that all recursive expressions can be enumerated and, hence, all recursive functions.

A. Church conjectured in 1936 that this class of functions was precisely the class of all effectively computable functions [**1**]. More accurately, to every effective rule for computing a sequence of natural numbers there exists a recursive expression with number e such that the function defined by the rule