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ON THE SINGULARITIES OF NONLINEAR FREDHOLM OPERATORS

BY M. S. BERGER[1] AND R. A. PLASTOCK

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Let f be a C^1 mapping between two Banach spaces X and Y. Then a key problem of functional analysis consists of describing the precise structure of the solutions of the equation $f(x) = y$, as y varies over Y under realistic hypotheses on the map f.

Motivated by the study of boundary value problems for systems of nonlinear elliptic partial differential equations we suppose the map f is a C^1 proper nonlinear Fredholm operator of nonnegative index acting between two *infinite dimensional* Banach spaces X and Y. Thus for each $x \in X$, $f'(x)$ is a linear Fredholm operator, index $f =$ index $f'(x)$ and $f^{-1}(K)$ is compact for each compact $K \subset Y$ (cf. Smale [7]).

In order to study the change in the structure of the solutions of $f(x) = y$ as y varies over y, we single out the singular points of f (i.e. the points x at which $f'(x)$ is not surjective). Known results study such mappings f with no singularities: for example, Banach and Mazur [1] assert that if f has index zero, f is a global homeomorphism; and a generalization to higher index by Earle and Eells [3].

Novel features of our results are their specifically infinite dimensional nature, the remarkable distinction they demonstrate between maps zero and positive Fredholm index, and their applicability to yield sharp results on the range of nonlinear elliptic systems (when combined with local bifurcation theory).

The main abstract results. Thus let f be a proper Fredholm nonlinear operator of index $p \geqslant 0$ as above. Let B denote the set of singular points of f and $S = f(B)$ denote the singular values of f. Then the following results hold:

THEOREM 1 (STRUCTURE THEOREM). *The solutions of $f(x) = y$ are either empty or homeomorphic compact p dimensional manifolds on each component of $Y - S$. In particular, for $p = 0$ the number of solutions of $f(x) = y$ are the same finite number on each component of $Y - S$.*

THEOREM 2 (REMOVABLE SINGULARITY THEOREM). *Suppose X and Y*

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are infinite dimensional separable Banach spaces and f has index zero. Then, if a singular point x_0 is isolated, f is a local homeomorphism about x_0. Moreover, if the singular points B are discrete, f is a global homeomorphism.

THEOREM 3 (ESSENTIAL SINGULARITY THEOREM). *Let f be a C^1 proper Fredholm operator of index $p > 0$. Then f has at least one singular point. Moreover, the singular set B cannot be locally compact and cannot have isolated points.*

These results are proven by:

(i) demonstrating that a proper nonlinear Fredholm operator f of nonnegative index p acting between X and Y defines a fibre bundle mapping between $X - f^{-1}(S)$ and $Y - S$ (where $S = f(B)$ denotes the singular values of f) provided $f(X) \neq S$.

(ii) utilizing a variety of results from algebraic and differential topology, and constructions of Church and Timourian [3].

For example, to prove the first part of Theorem 3 from (i), we begin by assuming that the operator $f: X \longrightarrow Y$ has no singular points. Then the exact homotopy sequence between the spaces X, Y and the fibre $M = f^{-1}(y)$ shows that the homotopy groups $\pi_n(M) = 0$ for $n \geqslant 1$. We show M is a connected compact p-dimensional orientable manifold and so by the Hurewicz theorem $\pi_p(M) \cong H_p(M) \cong Z$. This contradiction shows that f must have singular points. To prove the second part of Theorem 3, we use the fact that after excising a locally compact set Σ from the Banach space X, the sets X and $X - \Sigma$ are homeomorphic. The third part is proved by modifying a construction in [3].

Applications. 1. The stationary solutions of Navier-Stokes equations with appropriate inhomogeneous boundary conditions, forcing term and fixed, but arbitrary, Reynolds number R can be determined by the solutions of the equation $f_R(x) = y$. Here f_R is a C' proper Fredholm operator of index zero mapping a Sobolev space X of solenoidal vector fields into itself [4] and $y \in X$ is fixed. Thus, Theorem 1 implies off the singular values of f_R, the stationary solutions are finite in number, whereas Theorem 2 gives information about the stationary solutions at the onset of turbulence (cf. Foiaş and Temam [5]). In particular, Theorem 2 yields the new result that in many cases, at the lowest critical Reynolds number R_1, f_{R_1} is a local homeomorphism.

2. The equilibrium states of the combined bucking-bending problem for thin clamped elastic plates can be found by solving the associated nonlinear von Karman equations. Following Berger [6], these equations can be written in the form $A_\lambda(x) = y$, where A_λ is a C' proper nonlinear Fredholm operator of index 0 mapping the Sobolev space $X = \overset{\circ}{W}_{2,2}(\Omega)$ into itself and $y \in X$ is fixed. Let λ_1 and λ_2 denote the two smallest positive values of λ at which $A'_\lambda(0)x = 0$ has nontrivial solutions. Then for circular plates one can prove the following new sharp results: (i) For $\lambda \leqslant \lambda_1$, $A_\lambda(x)$ is a global homeomorphism; and (ii)

For $\lambda \in (\lambda_1, \lambda_2)$, the singular values of A_λ form a manifold of codimension 1 in X dividing X into connected components O_1 and O_3 on which the equation $A_\lambda(x) = y$ has exactly one or three solutions, respectively (joint work with P. Church).

ADDED IN PROOF (AUGUST 26, 1977). The multi-instanton solutions of Pontrjagin index k with gauge group G on S^4 of Euclidean field theory can be found by solving the nonlinear Yang-Mills equations. Recent work of Atiyah, Hitchin and Singer shows that $G = SU(z)$ the associated nonlinear (nonproper) operators are Fredholm of index $8k - 3$ with no singularities. Singularities do occur however for higher gauge groups.

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BELFER GRADUATE SCHOOL, YESHIVA UNIVERSITY, NEW YORK, NEW YORK 10033

DEPARTMENT OF MATHEMATICS, NEW JERSEY INSTITUTE OF TECHNOLOGY, NEWARK, NEW JERSEY, 07102

Current address (M. S. Berger): School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540