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THE EIGENVALUE SPECTRUM AS MODULI FOR COMPACT RIEMANN SURFACES

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The purpose of this note is to announce results concerning a moduli problem cited by I. M. Gel′fand. Suppose S is a compact Riemann surface of genus g, $g \geqslant 2$ endowed with the Poincaré metric and the associated Laplace-Beltrami operator Δ. The following question is considered: does the sequence of eigenvalues of Δ determine the conformal structure of S, [2], [3], [4], [5]. The Laplace-Beltrami operator is independent of the choice of orientation and similarly is the sequence of eigenvalues. In the case of genus one the answer is in the affirmative modulo the choice of orientation, [5]. We answer the question in the affirmative for genus greater than one except for a possible nongeneric situation. Details will appear elsewhere.

1. Preliminaries. Denote the Teichmüller space of genus g, $g \geqslant 2$ as T_g. Let M_g (resp. $\overline{M}_g$) be the Teichmüller modular group (resp. extended Teichmüller modular group) of genus g. A point $\{S\}$ of T_g is the equivalence class of a pair, a Riemann surface S of genus g and the homotopy class of a homeomorphism from a base surface S_0 to S. A point of T_g/M_g is the conformal equivalence class of a Riemann surface. A point of $T_g/\overline{M}_g$ is the conformal and anticonformal equivalence class of a Riemann surface. The reader is referred to [1] for a precise formulation.

Two fundamental sequences are associated with a compact Riemann surface S: the eigenvalue spectrum of Δ and the length spectrum the monotone sequence of lengths of all closed geodesics. By the technique of the Selberg trace formula the sequences determine each other [5], [8]. A Riemann surface S is represented as $\Gamma\backslash H$ where H is the upper half plane and Γ a Fuchsian group. Each closed geodesic of S corresponds to a conjugacy class in Γ; the length l and the common absolute trace t satisfy $t = 2\cosh(l/2)$.

2. Statement of results. The following theorem of H. P. McKean to our knowledge implies all prior results, [5]. A new proof is given.

THEOREM 1. *The total number of Riemann surfaces with a given length spectrum is finite.*

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The traces of properly chosen conjugacy classes of Γ are known to provide local coordinates for T_g. Given two surfaces with the same length spectrum a local map of T_g is defined in appropriate neighborhoods of points representing these surfaces. A group of real analytic automorphisms $\mathcal{M}_g$ is obtained from these local maps such that for $m \in \mathcal{M}_g$ and all $\{S\} \in T_g$, $\{S\}$ and $m(\{S\})$ have the same length spectrum. The considerations for $\mathcal{M}_g$ are summarized in the following.

THEOREM 2. *A real analytic subvariety* $\mathbf{V}_g$ *of* T_g *is defined. Given* $\{S\}$ *and* $\{R\}$ *elements of* T_g *having the same length spectrum then either* $m(\{S\}) = \{R\}$ *for some* $m \in \mathcal{M}_g$ *or* $\{S\}$ *and* $\{R\}$ *are both elements of* $\mathbf{V}_g$. *The index of* $\bar{M}_g$ *in* $\mathcal{M}_g$ *is finite.*

THEOREM 3. $\mathcal{M}_g = \bar{M}_g$.

These results are combined in the main theorem.

THEOREM 4. *Given* $\{S\}$ *a point of* $T_g - \mathbf{V}_g$ *then the conformal structure of S is uniquely determined modulo choice of orientation by the length spectrum of S.*

We do not know if $\mathbf{V}_g$ is nonempty.

An alternative description of Theorem 4 is obtained as follows. Let S be represented as $\Gamma\backslash H$. Consider Γ to be a discrete subgroup of $SL(2; \mathbf{R})$; the space $\Gamma\backslash SL(2; \mathbf{R})$ is a fiber space over $\Gamma\backslash H$. Given $f(x) \in L^2(\Gamma\backslash SL(2; \mathbf{R}))$ where L^2 is defined in terms of the invariant measure and $\gamma \in SL(2; \mathbf{R})$ we associate $f(x\gamma) \in L^2(\Gamma\backslash SL(2; \mathbf{R}))$. In this manner a representation of $SL(2; \mathbf{R})$, induced by the unit representation of Γ, in $L^2(\Gamma\backslash SL(2; \mathbf{R}))$ is obtained. This representation decomposes into a countable direct sum of irreducible representations of $SL(2; \mathbf{R})$. The duality theorem of [4] shows that this decomposition determines the eigenvalue spectrum.

THEOREM 5. *Let S and R be Riemann surfaces corresponding to points* $\{S\}$ *and* $\{R\}$ *of* T_g *with* $\{S\} \in T_g - \mathbf{V}_g$. *If the induced representations of S and R decompose into the same sum of irreducible representations, then S is conformally or anticonformally equivalent to R.*

3. Methods. A preliminary lemma estimates the variation in the length of the geodesic in a given free homotopy class under a quasiconformal deformation. Theorems 1 and 2 are obtained by combining this estimate with a theorem of D. Mumford [6]. The behaviour of the trace of a matrix under variation of the parameters which prescribe Γ is studied. A characterization of the first eigenvalue in terms of the shortest homologically trivial chain is employed, [7]. These considerations are combined with the technique of amalgamation of Fuchsian groups to yield Theorem 3.

REFERENCES

1. L. Bers, *Uniformization, moduli, and Kleinian groups*, Bull. London Math. Soc. 4 (1972), 257–300.

2. L. Ehrenpreis, *An eigenvalue problem for Riemann surfaces*, Ann. of Math. Studies 66 (1971), 131–140.

3. I. M. Gel'fand, *Automorphic functions and the theory of representations*, Proc. Internat. Congr. Math., Stockholm, 1962, pp. 74–85.

4. I. M. Gel'fand, M. I. Graev and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, W. B. Saunders, Philadelphia, 1969.

5. H. P. McKean, *Selberg's trace formula as applied to a compact Riemann surface*, Comm. Pure Appl. Math. 25 (1972), 225–246.

6. D. Mumford, *A remark on Mahler's compactness theorem*, Proc. Amer. Math. Soc. 28 (1971), 289–294.

7. R. Schoen, S. Wolpert and S. T. Yau, *Estimates for the first eigenvalue of a Riemann surface* (preprint).

8. A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. 20 (1956), 47–87.

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