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QUOTIENTS OF $C[0, 1]$ WITH SEPARABLE DUAL

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A. Pełczyński [3] and H. P. Rosenthal [5] considered conditions on a bounded linear operator T from $C(K)$, K compact metric, into an arbitrary Banach space which ensure that T is an isomorphism on a subspace of $C(K)$ isomorphic to a $C(S)$ space. (Throughout T will denote a bounded linear operator from $C(K)$, K compact metric, into a Banach space X.) Both used conditions on the "size" of $T^*B_{X^*}$ to produce their results. Pełczyński showed that if T is a nonweakly compact operator there is a subspace Y of $C(K)$ such that Y is isometric to c_0 and the restriction of T to Y is an isomorphism. Rosenthal's result is similar in form: If $T^*B_{X^*}$ is nonseparable then there is a subspace Y of $C(K)$ such that Y is isometric to $C(\Delta)$ (Δ is the Cantor set) and the restriction of T to Y is an isomorphism. Here we announce a similar result for $C_0(\omega^\omega)$, the space of continuous functions on the ordinals not greater than ω^ω and vanishing at ω^ω (the ordinals are considered in the order topology).

To state our result we need to recall the notion of index of a Banach space introduced by Szlenk [6].

DEFINITION. Let A be a bounded subset of a separable Banach space X and B be a bounded w^*-closed subset of X^*. For each $\epsilon > 0$ let $P_0(\epsilon, A, B) = B$ and define inductively a family of subsets of B indexed by the countable ordinals as follows: For each ordinal $\alpha < \omega_1$, $P_{\alpha+1} = \{b | \exists (b_n) \subset P_\alpha(\epsilon, A, B), b_n \xrightarrow{w^*} b$, and $\exists (a_n) \subset A, a_n \xrightarrow{w} 0$, such that $\overline{\lim}_{n\to\infty}\langle b_n, a_n\rangle \geq \epsilon\}$ and if β is a limit ordinal, let $P_\beta(\epsilon, A, B) = \bigcap_{\alpha<\beta} P_\alpha(\epsilon, A, B)$. The ϵ-Szlenk index of A and B is

$$\eta(\epsilon, A, B) = \sup\{\alpha: P_\alpha(\epsilon, A, B) \neq \emptyset\}.$$

Before we state our theorem let us note that since T is nonweakly compact if and only if $\eta(\epsilon, B_{C(K)}, T^*B_{X^*}) \geq 1$ for some $\epsilon > 0$, we can restate Pełczyński's theorem in terms of the index:

If for some $\epsilon > 0$, $\eta(\epsilon, B_{C(K)}, T^*B_{X^*}) \geq 1$, there is a subspace Y of $C(K)$ such that Y is isometric to c_0 and the restriction of T to Y is an isomorphism.

In this form our results are natural extensions of Pełczyński's.

THEOREM 1. *For every $k \in Z^+$ and $\epsilon > 0$ there is an integer $n(\epsilon, k)$ such*

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that if $\eta(\epsilon, B_{C(K)}, T^*B_{X^*}) \geqslant n(\epsilon, k)$, *there exists a subspace* Y_k *of* $C(K)$ *such that* Y_k *is isometric to* $C(\omega^k)$ *and the restriction of* T *to* Y_k *is an isomorphism with* $\|(T|_{Y_k})^{-1}\|$ *depending only on* ϵ.

The basic idea of the proof is to use the index condition to construct a subset F of $T^*B_{X^*}$ homeomorphic to the ordinals not greater than ω^k with the order topology. The definition of the index allows us to do this in such a way that the elements of F form a basic sequence equivalent to usual basis of l_1. From this an appropriate subspace Y_k of $C(K)$, isometric to $C(\omega^k)$, is constructed so that F norms Y_k in much the same way as the point mass measures norm $C(\omega^k)$.

If $\eta(\epsilon, B_{C(K)}, T^*B_{X^*}) \geqslant \omega$, we can combine subspaces Y_k derived from Theorem 1 to get

THEOREM 2. *If for some* $\epsilon > 0$, $\eta(\epsilon, B_{C(K)}, T^*B_{X^*}) \geqslant \omega$, *there is a subspace* Y *of* $C(K)$ *such that* Y *is isometric to* $C_0(\omega^\omega)$ *and the restriction of* T *to* Y *is an isomorphism.*

This result was recently used by M. Zippin in his solution of the separable extension problem. Since $\eta(1, B_{C_0(\omega^\omega)}, B_{C_0(\omega^\omega)^*}) = \omega$ and the index is an isomorphic invariant [6], the condition is necessary as well.

COROLLARY 1. *If* Y *is a complemented subspace of* $C(K)$, K *compact metric, and* Y *contains* $C(\omega^n)$ *uniformly, then* Y *contains a subspace isomorphic to* $C(\omega^\omega)$.

PROOF. Let P be the projection. Since Y contains $C(\omega^n)$ uniformly, $\eta(\epsilon, B_{C(K)}, P^*B_{Y^*}) \geqslant \omega$, for some $\epsilon > 0$.

COROLLARY 2. *The range of a projection* P *on* $C(\omega^\omega)$ *is isomorphic to* $C(\omega^\omega)$ *if and only if* $\eta(\epsilon, B_{C(\omega^\omega)}, P^*B_{C(\omega^\omega)^*}) \geqslant \omega$ *for some* $\epsilon > 0$.

PROOF. By Corollary 1, the range contains a subspace isomorphic to $C(\omega^\omega)$. From [4] it contains a complemented subspace isomorphic to $C(\omega^\omega)$, and hence, by the decomposition method [2] is isomorphic to $C(\omega^\omega)$.

Corollary 2 has been used by Y. Benyamini in his recent characterization of the complemented subspaces of $C(\omega^\omega)$ [1].

These results suggest the possibility of the following general theorem.

Let $\alpha < \omega_1$. If for some $\epsilon > 0$, $\eta(\epsilon, B_{C(K)}, T^*B_{X^*}) \geqslant \omega^\alpha$, then there is a subspace Y of $C(K)$ such that Y is isometric to $C_0(\omega^{\omega^\alpha})$ and the restriction of T to Y is an isomorphism.

Unfortunately, this is false for $\alpha = 2$. In fact we have constructed a bounded linear operator S from $C_0(\omega^{\omega^2})$ *onto* $C_0(\omega^{\omega^2})$ such that

$$\eta(½, B_{C_0(\omega^{\omega^2})}, S^*B_{C_0(\omega^{\omega^2})^*}) \geqslant \omega^2,$$

and for any subspace Y of $C_0(\omega^{\omega^2})$ which is isomorphic to $C_0(\omega^{\omega^2})$, the restriction of S to Y is not an isomorphism. The existence of such an operator is surprising since, if $C(K)$ is $C(\omega)$, $C(\omega^{\omega})$, or $C[0, 1]$, a bounded linear operator from $C(K)$ *onto* $C(K)$ must be an isomorphism when restricted to a suitably chosen subspace Y of $C(K)$ with Y isomorphic to $C(K)$. This follows from the result of Pełczyński, Theorem 2, and the result of Rosenthal, respectively.

The construction and verification of the example is lengthy. Details will appear elsewhere.

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