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CENTRAL SIMPLE ALGEBRAS WITH INVOLUTION

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We will carry the following hypotheses throughout this paper: F is a field of characteristic $\neq 2$; A is a *central simple* F-algebra, i.e. a simple F-algebra of finite dimension, with center F; A has an *involution* $(*)$ of *first kind*, i.e. an anti-automorphism of degree 2 which fixes the elements of F. The classic reference on central simple algebras is [1], which also treats involutions.

The dimension of A (over F) must be a perfect square, which we denote as n^2. A famous conjecture is that A must be a tensor product of a matrix subalgebra (over F) and quaternion subalgebras (over F); since the conjecture is easily proved when $n < 8$, the first case of interest is when $n = 8$. The main theorem of this paper is the following result when A is a division algebra.

MAIN THEOREM. *If $n = 8$, then A has a maximal subfield which is a Galois extension over F, with Galois group* $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

The proof relies heavily on a computational result of Rowen and Schild, which will be given below. Before sketching the proof of the main theorem, we start with some general results (true for any n), which can be verified easily.

PROPOSITION 1. *Given a subfield K of A containing F, we have an involution of A (of the first kind), which fixes the elements of K.*

PROPOSITION 2. *Suppose A is also a division algebra. Suppose K is a non-maximal subfield of A (containing F), with an automorphism φ over F, having degree 2. Then $\varphi^*: K \longrightarrow K^*$ can be given by conjugation in A, by an element which is symmetric (resp. antisymmetric) with respect to $(*)$.*

Let $\overline{F}$ denote the algebraic closure of F, and let $M_n(\overline{F})$ be the algebra of matrices over $\overline{F}$. Then $(*)$ induces an involution on $M_n(\overline{F}) \approx A \otimes_F \overline{F}$, given by $(\Sigma a_i \otimes \beta_i)^* = \Sigma a_i^* \otimes \beta_i$, for $a_i \in A$ and $\beta_i \in F$. We say $(*)$ is of *symplectic type* if the extension of $(*)$ to $M_n(\overline{F})$ is symplectic, i.e. *not* cogredient to the transpose (of matrices), cf. [1, p. 155]. Such an involution exists iff n is even, in which case we can build a "universal" F-algebra with symplectic type involu-

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tion, which we call $F(Y, Y^s)$. (See [3, §5] for details of construction, and for the properties of $F(Y, Y^s)$; here we write "F" in place of "Ω", which is used in [3].) Let F_1 = Cent $F(Y, Y^s)$, which is a central simple F_1-algebra of dimension n^2, with symplectic-type involution. By [3, Theorem 30], we can prove our main theorem by showing (when $n = 8$) that $F(Y, Y^s)$ has a maximal subfield which is a Galois extension over F_1, with Galois group $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Henceforth set $n = 8$. The key step to our main theorem is the following fact, which follows directly from a computation of Rowen and Schild [4] (done with the help of the IBM 370 computer at Bar Ilan University):

LEMMA 1. *$F(Y, Y^s)$ has an element x whose square is central.*

(The element x is in fact given by an explicit formula.) Now we sketch the proof of Theorem 1, working in $F(Y, Y^s)$. Using Proposition 1, we can find an involution (of first kind) under which x is symmetric; using Proposition 2, we can then modify the involution (*) such that (*) is symplectic and x is antisymmetric; i.e. $x^* = -x$. Then there is a symmetric element y, such that $yxy^{-1} = -x$. If $y^2 \in F_1(x)$ then y and x generate a quaternion F_1-subalgebra invariant under (*); in such a case, one concludes that $F(Y, Y^s)$ is a tensor product of quaternion subalgebras, and the theorem follows immediately. Thus we may assume $F_1(y^2) \cap F_1(x) = F$. Note $F_1(y^2) \neq F_1(y)$, and y (being symmetric) has degree 4 over F_1. Hence $[F_1(y^2): F] = 2$. We can then find z symmetric, such that $zxz^{-1} = x$ and $zy^2z^{-1} = -y^2 + \operatorname{tr}(y^2)/4$. If $z^2 \in F_1(x, y^2)$ then $z^2 \in F_1(y^2)$ (since z^2 is symmetric and x, xy^2 are antisymmetric); in this case y^2 and z generate a quaternion subalgebra invariant under (*), and again we are done. Thus, we may assume $F_1(z^2) \cap F_1(x, y^2) = 0$. Hence $F_1(x, y^2, z^2) = F_1(z)F_1(y^2)F_1(z^2)$, and the theorem follows immediately. Q.E.D.

One interesting aspect of this theorem is that $F(Y, Y^s)$ is used to produce a positive result. Previously, universal PI-algebras (without involution) had been used by Amitsur [2] to produce an important negative result.

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