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J. L. GASTWIRTH

Representations of real numbers by infinite series, János Galambos, Lecture Notes in Math., vol. 502, Springer-Verlag, Berlin, Hiedelberg, New York, 1976, vi + 146 pp., $7.40.

During the last thirty years a large amount of research has been devoted to the study of various algorithms for the representation of real numbers by means of sequences of integers. In addition to the two "classical" algorithms, i.e. digit expansions and continued fractions, and to several types of additional, slightly less well-known ones like Cantor, Lüroth, Engel and Oppenheim series, more general classes of such algorithms have been defined and investigated, particularly by F. Schweiger [6] and also by the author himself [3].

The book under review introduces the reader to some of the most important features of these developments. The exposition is based on what the author calls an (α,γ)-expansion $y(x)$ of a real number x. For each j, two strictly decreasing sequences $\alpha_j(n)$ and $\gamma_j(n)$ of positive real numbers are given, satisfying the condition

$$\alpha_j(n-1) - \alpha_j(n) \leqslant \gamma_j(n) \qquad (n = 2,3,\dots).$$

In order to define the algorithm for a given number $x \in (0,1]$, an auxiliary sequence $d_j(x)$ of integers is defined in such a way that the infinite series

$$y(x) = \alpha_1(d_1) + \gamma_1(d_1)\alpha_2(d_2) + \gamma_1(d_1)\gamma_1(d_2)\alpha_3(d_3) + \cdots$$

is always convergent and has, under fairly general assumptions, the limit x. A

criterion for the equation $y(x) = x$ is given, and it is shown that the classical
series expansions mentioned above fit into this pattern. The arithmetical part
of the theory is then concluded by an interesting chapter dealing with criteria
under which a number represented by a Cantor or Oppenheim series is
rational.

The main thrust of the book appears to lie in the metric and ergodic results
which it contains. After an introductory chapter on probability theory it is
shown how Cantor series, i.e. representations of the form

$$y(x) = \sum_{k=1}^{\infty} e_n(x)(q_1 q_2 \cdots q_k)^{-1}, \qquad 0 \leqslant e_k < q_k; q_k \geqslant 2,$$

fit into the general pattern of (α,γ)-expansions and that the "digits" $e_k(x)$,
considered as random variables on the unit interval, are stochastically inde-
pendent, something which is not true, e.g., for continued fractions.

The discussion of classical digit expansions is augmented by a section on
q-adic expansions with a nonintegral $q \in (1,2)$ satisfying an equation of the
form $q^{n+1} - q^n - 1 = 0$. This case merits particular interest inasmuch as the
digits, while not being independent, form a Markov chain in this case. The
author himself [1], [2] has, however, developed a method for reducing this
case to the case of independence.

In this context, i.e. in the class of algorithms with stochastically indepen-
dent digits, the author also discusses Lüroth series. They fit into the general
pattern of (α,γ)-expansions with $\alpha_j(n) = 1/n$ and $\gamma_j(n) = 1/n(n-1)$ for all j
and n.

In all these cases, the metric results proved concern asymptotic digit
properties which are valid for almost all x. Typical example (Theorem 4.6):
For a Cantor series algorithm with $\limsup q_n = +\infty$, *almost all x satisfy the
equation*

$$\limsup_{n \leqslant \infty} e_n(x)/q_n = 1.$$

In this connection, normality of digit expansions and, more generally, of
Cantor series is discussed.

Unfortunately, the concept of simple normality of a number x to the base q
(each digit occurs with the statistically expected asymptotic frequency) is
confused with normality (where this is required for all blocks of digits). Even
though this distinction is immaterial for the definition of absolute normality
(to be defined by either simple normality or normality of a number relative to
all bases $q = 2,3,\cdots$ simultaneously), the equivalence between these two
definitions is interesting and nontrivial in itself. (It can be deduced, e.g., from
a theorem of S. Pillai; see I. Niven [4, Chapter 8].)

In addition to a generalization of Borel's theorem on absolute normality to
Cantor series, a number of more recent metric results on the digits of Cantor
and Lüroth series are derived in neat form.

Here again, a careful reading of the text shows that some of the results on
nonnormal numbers are either quoted erroneously, or their logical relation to
each other is misinterpreted. For example, the author states (as Theorem 4.4)
the simple observation that, for any $z \in [0,1]$ and given q and a, there exists a

dense set of numbers $x \in [0,1]$ whose q-adic expansion involves the digit a with the asymptotic frequency z. This theorem is then attributed to T. Šalát [5] who, in reality, proves a deeper result concerning numbers whose digit frequencies fail to converge and have the entire interval [0,1] as set of limit points. In turn, the relation between this result of Šalát and results of the reviewer [7], [8] concerning joint distributions of all digits is misstated as if the latter ones could be obtained by the same argument.

The chapter on ergodic theory is placed relatively late in the book. Perhaps the exposition would be more fascinating to the average mathematician if, at an earlier stage, it would have been explained to him that, say, in the simple case of binary expansions, the shift operator $Tx = \{2x\}$ (which shifts the "decimal" point of x by one place to the right) turns out to be an ergodic mapping of the unit interval onto itself, preserving Lebesgue measure. The reader would then understand why ergodic theory became so useful to metric number theory during the last fifteen to twenty years and how Borel's theorem on normality was reproved by means of the individual ergodic theorem, using equivalence between normality of a number x and uniform distribution of the sequence $T^n x$ ($n = 0, 1, 2, \ldots$). Such an explanation might have served to motivate the author's exposition on applications of ergodic theory to Oppenheim series as one particular class of algorithms to which this approach can be generalized in some special cases, like Lüroth series, but not in others, like Sylvester series, where the corresponding shift operator fails to be ergodic.

The most general metric results of the book, presented in Chapter VI, again concern Oppenheim series, where a number of interesting asymptotic digit properties, including "laws of large numbers" which involve convergence in probability rather than convergence almost everywhere, are proved without using ergodic theory. The same is true for the chapter on Hausdorff dimension, where a good motivation of the concept and a fairly general "Eggleston type" theorem on the dimension of sets defined in terms of nested sets of intervals are given. But as its only application the author states the fact that the set of all x whose q-adic expansion does not involve $q - r$ specified digits, has dimension $\log r / \log q$. In particular, one might miss here some results on asymptotic digit expansions inasmuch as they augment Borel's theorem on normality.

The book closes with a list of ten well-chosen unsolved problems with explanations and references to existing partial results.

Naturally, a book of such a wide scope and such a short length could not be expected to be self-contained. Even though the basic facts on probability theory, ergodic theory and Hausdorff dimension are briefly stated, the reader will presumably need some prior knowledge of these fields in order to understand how they are applied. But any mathematically interested reader with these prerequisites will find the book stimulating and rewarding.

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BODO VOLKMANN

Ernst Edward Kummer, Collected Papers (edited by André Weil), Vol. I: *Contributions to number theory*, viii + 957 pp., $40.20, Vol. II: *Function theory, geometry and miscellaneous*, Springer-Verlag, Berlin-New York, 1975, xi + 877 pp., $42.20.

1. Whatever the reason for the eighty years' delay in the publication of Kummer's collected papers,[1] they now meet an audience whose interest in the number-theoretic contributions is, one expects, undiminished. If anything, the intervening years have given that audience a chance to catch its breath, and to absorb techniques ("*p*-adic"[2] in particular). The broad lines of Kummer's number-theoretic ideas now form an essential part of our heritage: it is fascinating to follow the details of their evolution.

The collected works are in two volumes. Volume I consists of Kummer's number theory. It constitutes a unity of thought and spirit almost from first sentence to last. One of the joys of reading it is in the double spectacle: the steady train of mathematical content, unimpeded by lack of basic algebraic number theory; while here and there, to serve problems at hand, the deft, unobtrusive forging of pieces of present day technique. It is not hard to get into, even for those of us who have had little contact with the history of our subject. Cleft though one may think one is from historical sources, on reading Kummer one finds that the rift is jumpable, the jump pleasurable. The reader is greatly helped in this jump in two ways. Firstly, included in the volume is a continuum of well-written, moving letters from Kummer to Kronecker giving the details of many of Kummer's important discoveries as they freshly occurred to him (these, together with some letters from Kummer to his

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[1] Weil, in his introduction, suggests that Hilbert, who dominated German mathematics in the late 19th and early 20th centuries, had little sympathy for Kummer's "*p*-adic" point of view, and asks whether this might not constitute a reason.

[2] Nowhere in Kummer's works will you find the word "*p*-adic" (nor, for that matter, "group"). The former term and concept were introduced far later by Hensel, but Kummer used the fact that formal power series in *p* may define numbers modulo p^n for arbitrarily high values of *n*. His progress seems so untrammelled by the lack of these explicit notions and so natural to describe by means of them that in the present review we will use modern language for them where suitable, although in some other respects, we shall try to be more faithful to Kummer's (explicit) ideas and choice of words.