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Ordinary differential equations in the complex domain, by Einar Hille, Pure and Applied Mathematics Series, John Wiley & Sons, New York, 1976, xi + 484 pp., $27.95.

The usual basic concepts and methods for ordinary differential equations in the complex domain are explained without going into tedious details. The reviewer believes that the readers will be able to familiarize themselves with those basics and that this book will be appreciated very much. It is fair to say that the interest of the author is more focussed on "Method" than "Intrinsic Meaning".

Through reading, the impression was that of listening to "Grandfather" while strolling with him in a quiet cemetary. He talks about good old days and beautiful people. In this book Lappo-Danilevskij is still alive, but Grothendieck does not exist. The introduction has two parts: Part I is "Algebraic and Geometric Structures" and Part II is "Analytical Structures". The contents of Part I actually belong to "Functional Analysis". They are not algebro-geometric in the sense of Grothendieck-Deligne-Katz (P. Deligne [3]). "Analytical Structures" means a collection of traditional basics for functions of one complex variable. The concept of analytic continuation is explained, but Riemann surfaces are not clearly defined. The resources look very meager. What can be accomplished? Indeed, not very much more than talking about various topics. The readers, however, would notice it whenever the author starts talking about something special. At such an instance, certain concepts and/or methods of higher quality are introduced. For example, in Chapter 4 the Nevanlinna theory is introduced. The author, in his 80's, still has a tremendous power of organizing research in this area.

The reviewer would cherish the contents of Chapter 8 very much as legacy from the author. This chapter contains also an open question which was posed by the author over 50 years ago (p. 291). Oscillation theory, in general, is of a highly technical nature. It is known that the author's contribution is eminent in this territory.

The author exhibits "Hypergeometric Euler Transformations" in §§7.3 and 7.4. For example, (7.3.20)–(7.3.23) on p. 262 show that the periods of an elliptic integral satisfy hypergeometric differential equations. Such a period can be written as $\int_{\gamma_z} \omega_z$ where ω_z is a 1-form on a Riemann surface X_z and γ_z is a 1-cycle on X_z. The triple $(X_z, \omega_z, \gamma_z)$, depending on a complex parameter z, is the prototype of the monster created by the algebro-geometric theory of linear differential equations with regular singularities (E. Brieskorn [**2**]). There is a global theory due to Ph. A. Griffiths [**7**], P. Deligne [**3**] and N. Katz [**10**], [**11**]. There is also a local theory due to E. Brieskorn [**1**]. The local theory is closely related with the asymptotic analysis of an integral of the form

$$\int_\gamma e^{\lambda f(t)} g(t)\, dt_1 \cdots dt_k,$$

where $t = (t_1, \ldots, t_k)$ is a variable in a domain X in C^k, $f(t)$ and $g(t)$ are analytic in X, γ is a singular k-chain in X and λ is a parameter (B. Malgrange [**13**]).

In the algebro-geometric theory, a system of linear differential equations is given by $\nabla u = 0$, where ∇ is the covariant differential defined by a completely integrable connection. This equation is, in general, a completely integrable Pfaffian system

$$\text{(S)} \qquad du = \left\{ \sum_{k=1}^{m} A_k(x)\, dx_k \right\} u,$$

where u is an n-vector, $x = (x_1, \ldots, x_m)$ is an independent variable in C^m and $A_1(x), \ldots, A_m(x)$ are n-by-n matrices whose components are functions of x. Set $\omega = \Sigma A_k(x)\, dx_k$. Then, system (S) is completely integrable if and only if $d\omega = \omega \wedge \omega$. System (S) is also given the form

$$\partial u / \partial x_k = A_k(x) u \qquad (k = 1, \ldots, m).$$

The algebro-geometric theory stimulates the study of Pfaffian systems in the complex domain.

The author of this book collected sufficiently many basics for ordinary differential equations. Therefore, it might be interesting for us to translate this book into another for Pfaffian systems. Certain results similar to that of Frobenius have been already obtained (R. Gérard and A. H. M. Levelt [**5**], M. Yoshida and K. Takano [**18**]). The reviewer would like to present here another "corny" result. (For the meaning of "corny", see E. Hille [**0**, p. 344].)

Consider a Pfaffian system

$$\text{(E)} \qquad x^{p+1}\frac{\partial u}{\partial x} = A(x,y)u, \qquad y^{q+1}\frac{\partial u}{\partial y} = B(x,y)u,$$

where u is an n-vector, (x, y) is a variable in C^2, p and q are *positive* integers, and $A(x, y)$ and $B(x, y)$ are n-by-n matrices whose components are holomorphic in a neighborhood of (0, 0). Suppose that system (E) is completely integrable and that $A(0, 0)$ and $B(0, 0)$ have *n distinct eigenvalues*, respectively. Then there exists an n-by-n matrix $P(x, y)$ such that (i) components of $P(x, y)$ are *holomorphic in a neighborhood of* (0, 0); (ii) $P(0, 0) \in \mathrm{GL}(n, C)$; (iii) the transformation $u = P(x, y)v$ *diagonalizes* system (E) (R. Gérard and Y. Sibuya [**6**]).

The global study of nonlinear Pfaffian systems leads us to a geometric theory of differential equations in the complex domain by means of complex analytic foliations. Among many activities in this direction are serious attempts to find intrinsic geometric meanings of classical results including those of P. Painlevé (R. Gérard [**4**]). The reviewer was told that it was Painlevé who originally introduced the concept of foliations into the study of differential equations (P. Painlevé [**14**]). The author of this book certainly cherishes the story of those days when Riemann surfaces and differential equations were getting along with each other very well. Such a rendezvous is now taking place in the algebro-geometric universe.

The reviewer would like to make a few comments on the Schwarzian (Chapter 10). The Schwarzian is also characterized in terms of connection. Such a characterization is given in algebro-geometric terms (P. Deligne [**3**, p. 33]). There is, however, another characterization which is closer to the contents of Chapter 10 (M. Schiffer and N. Hawley [**16**]). The author exhibits the following formula (10.1.9) on p. 376:

$$\{w, z\} = \{w, t\}(dt/dz)^2 + \{t, z\},$$

where $t = t(z)$ is a change of variable. Schiffer and Hawley [**16**] define the Schwarzian connection through this formula:

$$S_z(z)dz^2 = S_t(t)dt^2 + \{t, z\}\, dz^2$$

(Schiffer and Hawley [**16**, (14), p. 202]). The Schwarzian connection depends on a number of parameters. A conformal mapping and/or uniformization can be constructed by specifying values of those parameters. (Those parameters are called accessory parameters.) This guidance would enhance the appreciation of the readers who are interested in the contents of §10.2 (p. 377).

The result of Schwartz concerning algebraic solutions which enjoys an everlasting popularity is explained in §10.3 (p. 383). As the author points out, the crucial observation is that all finite subgroups of 2-by-2 matrices are known. It is reasonable to surmise that, if one has sufficient knowledge of finite subgroups of a group and sufficient knowledge of a monodromy group, then the program of Schwartz would work. (See, for example, K. Takano and E. Bannai [**17**].)

In 1950 and 1956, M. Hukuhara and S. Ohashi [**9**] obtained a class of Riemann's equations whose solutions are all expressible by the use of elementary functions and their integrals. In 1969, T. Kimura [12] showed that if solutions of Riemann's equation can be expressed in terms of elementary functions and their integrals, then the equation belongs to the class of either Schwartz or Hukuhara and Ohashi. Riemann's equations mean those differential equations whose general solutions are the Riemann P-function (p. 201). T. Kimura utilized E. R. Kolchin's "Picard-Vessiot theory" in his proof.

The readers who are interested in the contents of §10.3 would certainly enjoy reading the expository paper (on singularities) by E. Brieskorn [2].

The Schwarzian connection and quadratic differentials, and hence the theory of Teichmüller space, are intimately related. As a matter of fact, the contribution of Z. Nehari and the author is unique in this territory (§10.4, "Univalence and The Schwarzian", p. 388). As this romance developed, the notorious accessory parameters gained a few favor. In particular, see D. A. Hejhal [8].

The content of §10.5 (p. 394) was rescued by the author from oblivion (Preface, p. vii). Yet, *she* looks still sort of dizzy. The reviewer would like to recommend to the readers the little green book of E. G. C. Poole [15] as a reference for Chapter 10.

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Techniques of multivariate calculation, by Roger H. Farrell, Lecture Notes in Math., Springer-Verlag, New York, x + 337 pp., $12.30.

This book is primarily concerned with the mathematical techniques useful in calculating the distribution of functions of random matrices X: $n \times p$ where X has a multivariate normal distribution. As motivation for both this review and much of the material in Farrell's book, I will begin by posing a problem and discussing three possible approaches to solving it. Suppose X is an $n \times p$ random matrix ($n \geqslant p$) and X has a density $f(x)$ with respect to Lebesgue measure, l, on the linear space of $n \times p$ matrices. Let $S = X'X \equiv \tau(X) \in \mathcal{S}_p$, where $\mathcal{S}_p$ is the set of all $p \times p$ nonnegative definite matrices (S is positive definite a.e.). The problem is to find the density function of S.

APPROACH 1. Assume that the density $f(X)$ is a function of S as is the case when the elements of X are independent and normal with mean 0 and variance 1. Then $f(X) = g(X'X)$ for some function g. Hence, the density of S is $g(S)$ with respect to the measure $\mu \equiv l \circ \tau^{-1}$ on $\mathcal{S}_p$. All that remains is to calculate the measure μ. Wishart did this in 1928 using a geometric argument which led to the density bearing his name (in the normal case). Of course, $\mu(dS) = c|S|^{(n-p-1)/2}dS$ where c is a constant.

When f is not a function of $X'X$, then the above argument is not available. Two alternative approaches which can be used are now considered.

APPROACH 2. The group $\mathcal{O}(n)$ of $n \times n$ orthogonal matrices acts on the left of X by $X \to \Gamma X$, $\Gamma \in \mathcal{O}(n)$. A maximal invariant function under this action is $\tau(X) = X'X = S$. The density of S with respect to the measure μ given above is q where $q(\tau(x)) = \int f(\Gamma x)\nu(d\Gamma)$. Here, ν is the invariant probability measure on $\mathcal{O}(n)$. This result was used by James (1954) to derive an integral expression for the density function of the noncentral Wishart distribution in the rank 3 case. A result similar to the one above for general compact groups is due to Stein and will be discussed subsequently.

APPROACH 3. Let G_p denote the group of $p \times p$ upper triangular matrices with positive diagonal elements. Also, let $V_{n,p}$ be the set of $n \times p$ matrices ψ which satisfy $\psi'\psi = I_p$. $V_{n,p}$ is called the Stiefel manifold. Each X which has rank p (those with rank less than p have Lebesgue measure 0) can be uniquely written as $X = \psi U$ with $\psi \in V_{n,p}$ and $U \in G_p$. Since $S = X'X = U'U$, a method for finding the density of S is to first find the joint density of ψ and U and then "integrate out" ψ to yield the marginal density of U. With the density of U at hand, the derivation of the density of S is rather routine since the Jacobian of the map $S \leftrightarrow U$ ($S = U'U$) is easily calculated. To obtain the