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INVERSE SCATTERING FOR THE KLEIN-GORDON EQUATION

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In this note we would like to announce recent results concerning the so-called Inverse Scattering problem for the Klein-Gordon equation in three dimensions. Complete proofs of this work will appear in [1].

We consider the Klein-Gordon equation with a linear perturbation, that is

$$u_{tt} - \Delta u + m^2 u + q(x)u = 0 \tag{1}$$

in $\Omega = \mathbf{R}^3$, $-\infty < t < +\infty$. Here the subscripts denote partial derivatives, $m > 0$ and Δ is the Laplacian operator. The potential $q(x)$ is assumed to be a real valued function in $\mathbf{R}^3$, nonnegative and satisfying certain reasonable conditions at infinity which we will specify later. The initial Cauchy data for (1) at $t = 0$ will be assumed to be C^∞ with compact support. In the space of such solutions of (1) we define the (total) energy of u as

$$\|u\|_E^2 = \frac{1}{2}\int_{\mathbf{R}^3} [|\text{grad } u|^2 + u_t^2 + m^2u^2 + q(x)u^2]\, dx$$

where $|\text{grad } u|^2 = \Sigma_{j=1}^3 u_{x_j}^2$. It is easy to show that $\|u\|_E$ is constant i.e. we are dealing with a conservative equation. If we assume (for example) that $q(x) \in L^1 \cap L^\infty(\mathbf{R}^3)$ then it is well known (see for example [3] and [4]) that given a solution u of (1) there then exists a unique pair $u_\pm$ of solutions of (1) with $q \equiv 0$ such that

$$\|u - u_\pm\|_E \longrightarrow 0 \quad \text{as } t \longrightarrow \pm\infty.$$

The operator which relates $u_- \longrightarrow u_+$ is called the scattering operator and is denoted by S. One want to know what can be said about $q(x)$ if we know the operator S? This is a problem of physical relevance (see [5], [6]). If $q(x)$ is spherically symmetric, then there has been considerable research on this problem in the past twenty five years, mainly through the Gelfand-Levitan-Marchenko approach. In dimensions higher than one, very little is known. Here, we announce a "local" uniqueness result concerning the 3-dimensional inverse problem for (1).

THEOREM. *Let $q_1(x)$ and $q_2(x)$ be a nonnegative continuous functions which belong to $L^1 \cap L^\infty(\mathbf{R}^3)$. Let $S(q_1)$ and $S(q_2)$ denote the scattering operators associated with $u_{tt} - \Delta u + m^2u + q_1u = 0$ and $v_{tt} - \Delta v + m^2v + q_2v = 0$*

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respectively. If $S(q_1) = S(q_2)$, *then*

$$\lim_{\epsilon \to 0^+} \frac{\epsilon \|q_1 - q_2\|}{\alpha(\epsilon q_1, \epsilon q_2)} = 0.$$

Therefore, if $q_1(x) \neq q_2(x)$ *for some* $x \in \mathbf{R}^3$ *and the above limit is different from zero, then* $S(q_1) \neq S(q_2)$.

Here,

$$\|q_1 - q_2\| = \sup_{u_-} \sup_{\substack{x \in \mathbf{R}^3 \\ t \in \mathbf{R}}} \left| \int_{-\infty}^{\infty} R(x, t - r) * (q_1 - q_2) u_-(x, r)\, dr \right|$$

where u_- denotes any incoming free solution of (1) (with $q \equiv 0$), R the Riemann function of (1) with $q \equiv 0$, and $*$ denotes spatial convolution, $\alpha(q_1, q_2)$ is given by a constant times

$$(\|q_1\|_\infty^{1/3}\|q_1\|_1^{1/6} + \|q_1\|_1^{1/2}\|q_1\|_\infty^{1/2})(\|q_1\|_\infty^{1/3}\|q_1\|_1^{2/3} + \|q_1\|_1)$$

$$+ (\|q_2\|_\infty^{1/3}\|q_2\|_1^{1/6} + \|q_2\|_1^{1/2}\|q_2\|_\infty^{1/2})(\|q_2\|_\infty^{1/3}\|q_2\|_1^{2/3} + \|q_2\|_1).$$

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