mathematics undergraduate is not likely to have the knowledge of special functions, continuum mechanics, etc. needed to fully appreciate the applications. A number of interesting problems appear at the ends of the chapters, and the authors claim that the reader who evades the problems will miss 72% of the value of the book.

The problems treated in this text, as in the case of most books on partial differential equations, fall into the special category of well-posed problems. As pointed out by John in the book of Bers, John and Schechter, well-posed problems by no means exhaust the subject of partial differential equations. He observes that one may think of the solution of a well-posed problem as predicating the outcome of an experiment for a given arrangement of apparatus. The determination of what arrangement will produce a desired effect or what arrangement led to certain observed effects will correspond in general to a much more difficult mathematical problem, a problem that may not be well posed. In partial differential equations one may in fact not know, for a given equation, what classes of initial and/or boundary value problems are well posed, i.e., he may not know what apparatus to use.

A substantial percentage of interesting and important physical problems must of necessity be modeled as improperly posed mathematical problems, but since improperly posed problems can rarely be handled by standard analytical methods, such problems are largely ignored in books on partial differential equations.

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Topological transformation groups 1: *A categorical approach*, J. de Vries, Mathematical Centre Tracts, no. 65, Mathematisch Centrum, Amsterdam, 1975, v + 249 pp.

In this review, I will rapidly trace some stability concepts from their physical origins, along a path of increasing abstraction, into varieties of compact transformation groups. Much of the work already done (including de Vries' book) represents secondary technical research for which the primary investigations are still wanting.

I wish to thank Murray Eisenberg for helpful criticism and for the Markus and Palis references.

1. From differential equations to continuous flows. A 'nice' autonomous differential equation

$$\dot{x} = f(x), \qquad x \in X \subset \mathbf{R}^n,$$

admits unique solutions $\pi(x, t)$ with $\pi(x, 0) = x$, which are global in the sense that π is defined on $X \times \mathbf{R}$. If we refer only to the facts that X is a topological space and that $\pi: X \times \mathbf{R} \to X$ is a continuous action on X by the topological group $\mathbf{R}$ of reals, we may still discuss some of the qualitative dynamical properties of the system. This is where 'topological dynamics' comes from.

An arbitrary continuous group action $\pi: X \times \mathbf{R} \to X$ of $\mathbf{R}$ on the topological space X is called a *continuous flow*. Even if $X = \mathbf{R}^n$ it need not be true that a continuous flow is induced by a differential equation. See [Padulo and Arbib 1974, §2-2] for an introductory discussion. Some continuous flows find natural description without mentioning derivatives. One such example is 'billiard ball flows' (see [Birkhoff 1927, § VI.7] and [Birkhoff 1942]).

Consider the following hierarchy of stability concepts at a point x:

$$\text{asymptotically stable} \Rightarrow \text{stable} \Rightarrow \text{stationary}$$
$$\Rightarrow \text{periodic} \Rightarrow \text{almost periodic} \Rightarrow \text{Poisson stable}.$$

To elaborate, write $\mathfrak{N}_x$ for the neighborhood filter of x; write $\mathbf{R}^+$ for $\{t: t \geqslant 0\}$; write xt for $\pi(x, t)$; for $A \subset \mathbf{R}$, $xA = \{xt: t \in A\}$; $L^+(x)$ denotes the *positive limit set* of x, $L^+(x) = \{y \in X: \exists t_n \text{ with } t_{n+1} > t_n \geqslant n, y = \lim xt_n\}$; $A \subset \mathbf{R}$ is *relatively dense* if 'gaps are bounded', that is, for some $M > 0$, every interval of length M intersects A. Then

x is *stationary* if $\forall t$, $xt = x$.

x is (*Poincaré* or *Lyapunov*) *stable* if x is stationary and if $\forall U \in \mathfrak{N}_x$ $\exists V \in \mathfrak{N}_x$ with $V\mathbf{R}^+ \subset U$.

x is *asymptotically stable* if x is stable and if $\exists U \in \mathfrak{N}_x$ $\forall y \in U$, $x \in L^+(y)$.

x is *periodic* if $\exists \theta > 0$ $\forall t$, $xt = x(t + \theta)$; the least such θ is the *period* of x.

x is *almost periodic* if $\forall U \in \mathfrak{N}_x$, $\{t \in \mathbf{R}: xt \in U\}$ is relatively dense.

x is *Poisson stable* if $\forall U \in \mathfrak{N}_x$, $\{t > 0: xt \in U\}$ and $\{t < 0: xt \in U\}$ are unbounded.

A subset A of X is *invariant* if $At \subset A$ for all t. For any x, the *orbit closure* $\mathrm{cls}(x\mathbf{R})$ and $L^+(x)$ are closed, invariant sets. An inclusion-minimal nonempty closed invariant set is called a *minimal set*.

For the continuous flow $+: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ (induced by $\dot{x} = 1$) no point is Poisson stable, but $x\mathbf{R}$ is minimal for every x. On the other hand, *compact* minimal sets enjoy stability properties as was first observed by [G. D. Birkhoff 1912] who proved that if X is a {compact} {complete metric} space then $\mathrm{cls}(x\mathbf{R})$ is a compact minimal set {if and only if} {if} x is almost periodic. Thus for, say, the flow on $\mathbf{R}^n$ induced by a differential equation, it is important to consider compact invariant subsets.

For a more detailed historical sketch see [Sell 1976]. For further discussion see [Bhatia and Szegö 1970], [Cartwright 1964], [Hájek 1968], [Nemytskii and Stepanov 1960], [Sell 1971] and the bibliographies there.

2. From continuous flows to topological dynamics. A *topological transformation group* is a continuous action $\pi: X \times T \to X$ of a topological group T on a topological space X. The special case $T = \mathbf{R}$ recovers continuous flows. A continuous trajectory $\pi(x, -)$ admits a 'step-approximation'

$$n \mapsto \pi(x, n \cdot \Delta t)$$

for n an integer and Δt fixed (and small). giving rise to the concept of a *discrete flow* where $T = \mathbf{Z}$, the discrete group of integers. In *differentiable dynamics*, X is a smooth manifold, T is a Lie group and π is smooth; see [Bredon 1972], [Brockett], [Smale 1967] and their bibliographies.

Given an arbitrary topological transformation group (X, T, π) and an element x of X,

x is *stationary* if $\forall t \in T\ xt = x$.

x is *periodic* if $\exists \theta \neq e$ with $x = x\theta$; the subgroup of such θ is the *period* of x.

It is a little more challenging to generalize 'almost periodic' and 'Poisson stable'. Say that $A \subset T$ is *syndetic* if $AK = T$ for some compact subset K of T. For $T = \mathbf{R}$, syndetic is equivalent to relatively dense. Dense implies syndetic if T is locally compact. Also, say that $A \subset T$ is *replete* if $\forall$ compact K in T $\exists t_1, t_2 \in T$ with $t_1 K t_2 \subset A$. Then

x is *almost periodic* if $\forall U \in \mathfrak{N}_x$, $\{t \in T\colon xt \in U\}$ is syndetic.

x is *recurrent* if $\forall U \in \mathfrak{N}_x$, $\{t \in T\colon xt \in U\}$ intersects every replete subsemigroup of T.

For $T = \mathbf{R}$, recurrent is equivalent to Poisson stable. As before, 'stationary' implies 'periodic' implies 'almost periodic' implies 'recurrent', although the last implication assumes local compactness of X [Gottschalk and Hedlund 1955, 7.05].

If S is a subgroup of T, the restriction of π induces a topological transformation group $X \times S \to X$. The motivating example is $\mathbf{Z} \subset \mathbf{R}$ which may be abstracted by observing that $\mathbf{Z}$ is a closed syndetic normal subgroup of $\mathbf{R}$. Of particular vitality in topological dynamics are those concepts which appear the same when viewed in either the discrete or continuous context, that is, which hold if and only if they hold under a fixed, but arbitrary, syndetic normal subgroup. 'Stationary' and 'periodic' behave poorly. To see this, consider a point x on an orbit of period 1 for a simple harmonic oscillator and observe that x is stationary under $\mathbf{Z}$ but is not periodic under $\alpha\mathbf{Z}$ if α is irrational. But such x is at least almost periodic under $\alpha\mathbf{Z}$. In fact, the following two 'inheritance theorems' (with a common proof–see [Gottschalk and Hedlund 1955, 3.36]) hold:

Let T be locally compact, let S be a closed syndetic normal subgroup of T and let $x \in X$. Then x is {almost periodic} {recurrent} under T if and only if x is {almost periodic} {recurrent} under S.

Birkhoff's theorem also generalizes [Gottschalk and Hedlund 1955, 4.05, 4.07]:

If X is a compact minimal set, every element of X is almost periodic. If X is regular and x is almost periodic, $\operatorname{cls}(xT)$ *is a minimal set.*

The classification of compact minimal sets is a central research problem in topological dynamics. For further discussion see [Auslander, Green and Hahn 1963], [Ellis 1969], [Gottschalk 1958, 1963, 1964], [Gottschalk and Hedlund 1955], and [Montgomery and Zippin 1955]. For an extensive bibliography see [Gottschalk 1966]. For a discussion of 'Lyapunov stability vs. uniform almost periodicity' see [Sell 1971, p. 106].

3. Emoting. I think it is possible to write a good book on mainstream topological dynamics which includes sensitive applications of principles from category theory. I do not think de Vries' book has achieved either objective. In writing this review, I hope to make clear some of the issues that the would-be author needs to know about. This section is devoted to emotional issues.

De Vries' book contains a quarter of a thousand pages and has sections entitled 'preliminaries' and 'generalities'. On p. 1 it claims to be a self-contained treatise. Quoting from the preface and p. 32 we find

> "We intend to lead the reader over a more or less artificial path between a number of 'vantage points' [of the theory of topological transformation groups] . . . "

> "We shall not enter into the history and the development of the concept of a ttg. Nor shall we try to convince the reader of the importance of ttgs."

The book moves from one detailed topic to the next with no overview. I can only wonder for whom this book was written (no help there from the author). The would-be author needs to include history and overview, needs to provide discussion of and pointers to the mainstreams and needs to avoid artificial paths. (The 'notes' in the book *are* interesting and well researched. De Vries is well qualified as a would-be author as I hope will be evident in *Topological transformation groups* 2.)

The thesis of the book may be inferred from two quotes from the preface:

> " . . . we have chosen to take a mainly categorical point of view, with the aim of unifying parts of the existing theory of ttgs."

> " . . . facts about a certain category of ttgs should be expressed in terms of the underlying categories of topological groups and topological spaces. (Although this tactic will probably hurt the feelings of every pure category theorist, it is a consequence of their wish that each 'working mathematician' should try to use category theory for the description of the objects he is studying.)"

This thesis is neither motivated nor discussed. It is simply amply done. As such, de Vries' book will further polarize opponents of category theory; this problem, discussed further below, needs to be addressed head on by the would-be author.

My colleague Michael Arbib points out that the textbook title *Control theory with matrices* is old-fashioned in 1976. I hope the time will come when I can feel the same way about 'the categorical point of view'. A *mainly* categorical point of view is rather overbearing in topological dynamics. In TTG 2, de Vries needs to explain why TTG 1 is relevant.

The comment about the feelings and wishes of us category theorists, pure or impure, I will let pass by save to mention that the second sentence of [Mac Lane 1971] has been brutally misrepresented.

Category theory offers a few basic results (for example, the adjoint functor theorems, the existence of Kan extensions, the construction of limits of arbitrary diagrams from products and equalizers) whose incisive use in context establishes healthy directions of inquiry, albeit without providing the desired depth of analysis. In other words, substantial applications of category theory will not come from people who know only category theory. It is ironic that the strongest opponents of category theory are often those most qualified to make use of it. At the risk of being brash, I refer the reader to [Manes 1974] for what I mean by an application of category theory. A function space construction is given which is sufficiently concrete to allow the design of examples with prescribed properties and which yields precisely the class of dual Banach spaces. I challenge the reader to prove this without using the special adjoint functor theorem (which is an unexplored and fascinating general technique to construct Banach spaces).

Many areas of mathematics (not just category theory) have branched in directions of research which are disconnected from their original motivations. This applies to topological dynamics. Even the basic idea of a continuous flow is restrictive since, for example, the flows $xt = \pm(t + x^2)^{1/2}$ induced by the scalar equation $2x\dot{x} = 1$ are not global. (Some authors replace $\dot{x} = f(x)$ with the 'equivalent' global system $\dot{x} = f(x)/(1 + \|f(x)\|)$, but equivalence is only in the sense of 'same trajectories'; what physicist is willing to reformulate Newton's Law as $F = GMm/(r^2(1 + \mathrm{r}))$?) (However, see [Markus 1971] for the role of homeomorphisms which preserve sensed orbits in questions of structural stability.) One can discuss local flows *per se* (see e.g. [Sell 1971] and [Hájek 1968]). For another example, witness that compact minimal sets (see e.g. [Ellis 1965], [Floyd 1949], [Gottschalk 1963]) are often abstract discrete flows for which $x \mapsto x1$ is not of the form $\pi(-, 1)$ for any continuous action π (e.g., $x \mapsto x1$ may not be homotopic to the identity).

Similar problems occur in the differentiable case [Palis 1974]. Really gutsy questions such as 'Which compact minimal discrete flows arise as $\mathrm{cls}(x\mathbf{N})$ under the flow induced by a differential equation on a manifold?' have so far been too difficult to answer.

We began with a differential equation and have, so far, ended up with a compact minimal set. Something was lost. The would-be author must not think that 'compact minimal sets = abstract qualitative theory of differential equations' but should be aware of the relationships and the history. In the remainder of the review we enter some of the heartland of topological dynamics with one ear open for categorical echoes.

4. Function spaces. Usually, stability properties of x hold for all y in xT and become properties of the motion $\pi(x, -)$ in the set $C(T, X)$ of continuous functions. Let $C_c(T, X)$ denote the compact-open topology. If T is locally compact separated then

$$C_c(T, X) \times T \xrightarrow{L} C_c(T, X)\colon (f, t) \mapsto fL_t$$

(where $L_t(s) = ts$) is a continuous action, and the passage $x \mapsto \pi(x, -)$ is a continuous equivariant injection. $C_c(\mathbf{R}, \mathbf{R})$ is the *Bebutov dynamical system* [Bhatia and Szegö 1970], [Sell 1971]. It is not hard to see that x is almost periodic in (X, π) if and only if $\pi(x, -)$ is almost periodic in $(C_c(T, X), L)$. Similarly, x is *Lagrange stable* in (X, π) (that is, $\mathrm{cls}(xT)$ is compact) if and only if $\pi(x, -)$ is Lagrange stable in $(C_c(T, X), L)$.

If X is a complete uniform space, if $C_u(T, X)$ denotes the topology of uniform convergence, and if $|T|$ denotes T with the discrete topology (*qua* group or set, depending on context), then

$$C_u(T, X) \times |T| \to C_u(T, X)\colon (f, t) \mapsto fL_t$$

is a continuous action. Lagrange stable points in $C_u(T, X)$ are called *von Neumann almost periodic functions* [von Neumann 1934]. The almost periodic points of $C_u(T, X)$ are called *Bohr almost periodic functions* [Bohr 1947]. For $T = \mathbf{R}$ and $X = \mathbf{C}$, Bohr characterized such functions as the uniform limits of trigonometric polynomials. In §2.2, de Vries provides proofs that von Neumann almost periodic implies Bohr almost periodic (and conversely, if T is abelian); this is unlike the situation encountered with continuous flows wherein Lagrange stable is implied by almost periodic, but not conversely in general.

We may sum up by saying that $C_c(T, X)$ is a natural object if one is interested in 'X under some flow' whereas $C_u(T, X)$ has firm roots in harmonic analysis. For a clearer perspective on the role of Bohr almost periodic functions in the theory of differential equations, see [Sell 1976] and [Fink 1974].

5. Homomorphisms. A *homomorphism* (f, ψ): $(X, S, \pi) \to (Y, T, \rho)$ is a continuous group homomorphism ψ: $S \to T$ and a continuous map f: $X \to Y$ subject to the equivariance condition $\rho(f(x), \psi(s)) = f\pi(x, s)$. Following de Vries, let **TTG** denote the resulting category and let $\mathbf{Top}^T$ be the subcategory '$\psi = \mathrm{id}_T$'. De Vries also considers the category $\mathbf{TTG}_*$ of (f, ψ): $(X, S, \pi) \to (Y, T, \rho)$ where ψ: $T \to S$ and $\rho(f(x), t) = f\pi(x, \psi(t))$. About half of de Vries' book is about these categories and their Kelley space [Kelley 1955, p. 230] variants with regard to internal properties such as completeness and external properties such as nice functors to simpler categories. This is a secondary investigation. A primary question is whether homomorphisms of dynamical systems contain information or are interesting. I think they do and they are. Not being aware of suitable references, I will record a few examples of the simplest kind. I hope that the results recorded in de Vries' book eventually find applications stemming from investigations of the sort hinted at by these examples.

EXAMPLE 1. Let A: $\mathbf{R}^n \to \mathbf{R}^n$ be a matrix with Jordan form $J = PAP^{-1}$. Then $\pi(x, t) = e^{At}x$, $\rho(y, t) = e^{Jt}y$ are the flows corresponding to $\dot{x} = Ax$. $\dot{y} = Jy$. If P is real, e^P: $(\mathbf{R}^n, \pi) \to (\mathbf{R}^n, \rho)$ is an isomorphism in $\mathbf{Top}^{\mathbf{R}}$.

EXAMPLE 2. 'Rolling a wheel at constant speed' is a homomorphism in $\mathbf{Top}^{\mathbf{R}}$. Consider $\dot{x} = 1$ on $\mathbf{R}$ with flow $\pi(x, t) = x + t$, let $\mathbf{S}^1$ be the unit circumference, and consider the flow $\rho(y, t) = y + t$ induced by the unit tangent vectorfield. Then the mod 1 projection f: $(\mathbf{R}, \pi) \to (\mathbf{S}^1, \rho)$ is a homomorphism.

EXAMPLE 3. The simple harmonic oscillator $\ddot{x} = -K^2x$ induces the continuous flow

$$\pi_K(x, y) = (x \cos(Kt) + (y/K)\sin(Kt), -Kx \sin(Kt) + y \cos(Kt))$$

on $\mathbf{R}^2$ (x = position, $y = \dot{x}$). If $K > L > 0$, (R^2, π_K) admits no homomorphism to (R^2, π_L) in $\mathbf{Top}^{\mathbf{R}}$, but these flows are isomorphic in **TTG** via $\psi(t) = (K/L)t, f(x, y) = (x, (L/K)y)$. For each $r \neq 0$, $g_r(x, y) = (rx, ry)$ is an automorphism of $(\mathbf{R}^2, \pi_K)$ in $\mathbf{Top}^{\mathbf{R}}$.

EXAMPLE 4. $\mathbf{Top}^T$ has products [Mac Lane 1971, III.4] in the obvious way (t acts independently on each coordinate). Consider $(\mathbf{R}, +)$ as in Example 2. If (X, π) is a continuous flow, the trajectories of $(X, \pi) \times (\mathbf{R}, +)$ are the solution curves of (X, π). 'Projection' $(X, \pi) \times (\mathbf{R}, +) \to (X, \pi)$ is a homomorphism in $\mathbf{Top}^{\mathbf{R}}$.

EXAMPLE 5. Consider the nonautonomous scalar equation $\dot{x} = 2tx$. Then if $\varphi(x, t) = x \exp(t^2)$, $\varphi(x, -)$ is a solution with $\varphi(x, 0) = x$, but φ is not a flow. The associated autonomous system is $(x, u)^{\cdot} = (2ux, 1)$ with flow $\pi(x, u; t) = (x \exp(-u^2) \exp((u + t)^2), u + t)$. Here $(\mathbf{R}^2, \pi)$ is not isomorphic to $(\mathbf{R}, \rho) \times (\mathbf{R}, +)$ for any ρ. This flow is *reversible* in the sense that $\psi(t) = -t, f(x, u) = (x, -u)$ is an automorphism in **TTG.** In general, if T is abelian, $\pi(-, u_0)$: $(X, \pi) \to (X, \pi)$ is an automorphism ('change of time origin') in $\mathbf{Top}^{\mathbf{T}}$. Thus for the flow above,

$$f(x, u) = \big(x \exp(-u^2) \exp\big((u - u_0)^2\big), u - u_0\big)$$

is an automorphism.

EXAMPLE 6. De Vries gives the following motivation for considering $\mathbf{TTG}_*$. Given (Y, T, ρ) and $XT \subset X \subset Y$, $N = \{t \in T\colon xt = x \text{ for all } x \text{ in } X\}$ is a normal subgroup of T. If $f\colon X \to Y$ is the inclusion map and if $\psi\colon T \to T/N$ is the canonical surjection, (f, ψ) is a morphism in $\mathbf{TTG}_*$. In the context of Example 2 with $X = Y$, T/N is the circle group.

Homomorphisms often preserve stability properties. Given $(f, \psi)\colon (X, S, \pi) \to (Y, T, \rho)$ in **TTG** and x in X, $f(x)$ is periodic if x is; $f(x)$ is Lagrange stable if x is (side condition: Y is Hausdorff); $f(x)$ is {almost periodic} {recurrent} if x is (side condition: ψ is onto); and, (Y, T, ρ) is minimal if (X, S, π) is, providing f and ψ are onto. Morphisms in $\mathbf{TTG}_*$ often fail to preserve stability properties.

6. Varieties. In this section, X is always compact Hausdorff.

In [Manes 1976, § 4.1] compact transformation groups are dealt with as 'sets with algebraic structure', that is, the full subcategory $\mathbf{Comp}^T$ of $\mathbf{Top}^T$ (X is compact Hausdorff) is 'monadic over **Set**' where **Set** is the category of sets and functions. In other words, a compact transformation group may be regarded as a set equipped with certain operations subject to certain equations without loss of information (the topology is included). This produces, in effect, a pleasant rapprochement between the ideas of G. D. Birkhoff (a major figure in founding the formal theory of dynamical systems) and his son Garrett (who in [Birkhoff 1935] founded universal algebra). Since the exposition in [Manes 1976] proceeds from first principles we will avoid an exposition of 'monad theory' here, preferring to describe this point of view of $\mathbf{Comp}^T$ in more standard language.

$$\begin{tikzcd} A \arrow[rr, "\eta_A"] \arrow[dr, "f"'] & & V(A) \arrow[dl, dashed, "f^{\#}"] \\ & (X, \pi) & \end{tikzcd}$$

Let T be discrete. For any set A, T acts on $A \times T$ via $(a, t)s = (a, ts)$ so that the β-compactification $\beta(A \times T)$ (viewed as the set of ultrafilters on $A \times T$) equipped with the continuous extensions of $(a, t) \mapsto (a, ts)$ $(s \in S)$ is in $\mathbf{Comp}^T$. In fact, $\beta(A \times T)$ is *freely generated by* A in the sense that, as shown in the diagram, any function f from A to the underlying set of a compact transformation group admits a unique continuous equivariant extension $f^{\#}$; specifically, $V(A) = \beta(A \times T)$, $(\mathfrak{N}_A)(a)$ is the principal ultrafilter on (a, e) and for $\mathfrak{U} \in \beta(A \times T), f^{\#}(\mathfrak{U})$ is the convergent point of

$$\left\{B \subset X: f^{-1}\left\{(a, t) \in A \times T: \pi(f(a), t) \in B\right\} \in \mathfrak{U}\right\} \in \beta|X|.$$

More generally, let **V** be any *variety* in $\mathbf{Comp}^{|T|}$, that is, a full subcategory closed under products, subobjects (i.e., injective homomorphisms into) and quotients (i.e., surjective homomorphisms out of). Then, using the general adjoint functor theorem [Mac Lane 1971, V.6], every set A freely generates an object $V(A)$ of **V** (see the diagram). $V(A)$ is a quotient of $\beta(A \times T)$ and $f^{\#}$ is well defined on equivalence classes, but this is as close as one comes to a construction of $V(A)$ in general. $\mathbf{Comp}^T$ is a variety in $\mathbf{Comp}^{|T|}$. For any class **A** in $\mathbf{Comp}^{|T|}$ the variety Var(**A**) generated by **A** coincides with the class of all quotients of subobjects of products of elements of **A**.

Given (X, π) in a variety **V**, if A is a subset of X with inclusion function i: $A \to X$, the subobject $\langle A \rangle = \mathrm{cls}(AT)$ generated by A is the image of $i^{\#}$: $V(A) \to (X, \pi)$. Let us call a minimal (X, π) which happens to be in **V** a **V**-*minimal set*. Every **V**-minimal set is a quotient of $V(1)$ (where 1 is a one-element set). Under the binary operation

$$V(1) \times V(1) \to V(1): (p, q) \mapsto 1 \xrightarrow{p} V(1) \xrightarrow{q^{\#}} V(1),$$

$V(1)$ is a monoid. More generally, $V(1)$ acts on any (X, π) in **V** via

$$|X| \times V(1) \to |X|: (x, p) \mapsto \left(1 \xrightarrow{x^{\#}} X\right)(p).$$

Notice that $\langle x \rangle$ is the orbit of x under $V(1)$ for all x in X so that, in some sense, a continuous group action has been converted into a discrete monoid action.

The *enveloping semigroup* $E(X, \pi)$ of (X, π) [Ellis 1960a, 1969] is $\langle \mathrm{id}_X \rangle \subset (X, \pi)^{|X|}$, that is, $E(X, \pi)$ is the pointwise closure of $\{\pi(-, t): t \in T\}$. Then $E(X, \pi)$ is just $V(1)$ if $\mathbf{V} = \mathrm{Var}(X, \pi)$. This makes contact with the tradition in universal algebra of studying an algebra in terms of the variety it generates (i.e., its equational theory), and suggests the importance of $E(X, \pi)$. Even when $T = \mathbf{R}$, $E(X, \pi)$ can fail to be a group and can fail to be commutative.

A *universal* **V**-*minimal set* is a **V**-minimal set admitting at least one homomorphism (necessarily surjective) to every **V**-minimal set. Say that (X, π) is *coalescent* if each of its endomorphisms is an automorphism. Every minimal subset of $V(1)$ (such exists by Zorn's lemma) is coalescent, hence is the unique-up-to-isomorphism universal **V**-minimal set. Thus the universal **V**-minimal set exists uniquely and is a monoid acting on every object in **V**. The universal **V**-minimal set has a strictly monoid-theoretic characterization: it is any minimal right ideal of the monoid $V(1)$ (see [Ellis 1960b]). In particular, any minimal right ideal of $E(X, \pi)$ provides the universal minimal set of $\mathrm{Var}(X, \pi)$. Also, for example, there exists a fixed minimal continuous flow whose quotients form precisely the class of all minimal continuous flows.

The general principle here is 'properties which stay closed under products, subobjects and quotients admit universal objects'.

We close this section by discussing two varieties which have received much attention in the literature.

The *proximal relation* $P(X, \pi)$ of (X, π) consists of all pairs (x, y) which are *proximal* in the sense that for every entourage (neighborhood of the diagonal) α there exists t in T with (xt, yt) in α. By [Auslander 1963], (X, π) is the universal minimal set of $\mathrm{Var}(X, \pi)$ if and only if (X, π) is minimal, and for all x, y there exists $f: (X, \pi) \to (X, \pi)$ with $(f(x), y) \in P(X, \pi)$.

(X, π) is *distal* if $P(X, \pi)$ is the equality relation. Some equivalent characterizations [Ellis 1969, Chapter 5]: (a) every function in $E(X, \pi)$ is injective; (b) every function in $E(X, \pi)$ is bijective; (c) every point in $(X, \pi) \times (X, \pi)$ is almost periodic. In [Ellis 1958, p. 405], Ellis credits Hilbert with using 'distal' to topologically characterize the concept of a rigid group of motions. It is known [Wu 1968] that if (X, π) is minimal, distal and metrizable then (X, π) is coalescent. If T is abelian and if (X, π) is minimal but not distal, $P(X, \pi)$ is neither closed nor an equivalence relation [Keynes 1967].

(X, π) is *equicontinuous* if $\{\pi(-, t): t \in T\}$ is equicontinuous. Equivalent characterizations [Ellis 1969, Chapter 4]: (a) (X, π) is *uniformly almost periodic*, that is, for all entourages α there exists syndetic $A \subset T$ such that $xA \subset x\alpha$ for all x; (b) $E(X, \pi)$ is a compact topological group of homeomorphisms of X.

'Distal' and 'equicontinuous' are varieties; denote them **Dist**, **Eq**. Thus **Eq** $\subset$ **Dist**, and, for (X, π) distal, each point is almost periodic. Minimal, distal, nonequicontinuous differentiable flows on compact homogeneous spaces of nilpotent Lie groups are constructed in [Auslander, Hahn and Markus 1963]. Any intersection of varieties is a variety. The $\{$*distal*$\}$ $\{$*equicontinuous*$\}$ *structure group* of (X, π) [Ellis and Gottschalk 1960] is the monoid $V(1)$ in the variety $\{\mathrm{Var}(X, \pi) \cap \mathbf{Dist}\}$ $\{\mathrm{Var}(X, \pi) \cap \mathbf{Eq}\}$. We observe that whenever $(X, \pi) \in \mathrm{Var}(Y, \rho)$ that $E(X, \pi)$ is a simultaneous monoid and transformation group quotient of $E(Y, \rho)$ (consider the Ith component of the corresponding theory map [Manes 1976, §3.2]). This explains why $V(1)$ above is a group.

7. Kan what may. The problem of associating a continuous flow to an abstract discrete flow is well motivated and it is natural, therefore, to ask about adjoints of the functor $\mathbf{Top}^{\mathbf{R}} \to \mathbf{Top}^{\mathbf{Z}}$ of §2. Using a very general principle concerning Kan extensions [Mac Lane 1971, Theorem X.3.1] (with a little extra *ad hoc* work owing to the group topologies) it is immediate that this functor has left and right adjoints. The left adjoint is well known [Gottschalk 1973, p. 123] and is discussed in de Vries' book as 3.3.8, 5.3.6. The right adjoint seems not to be in the literature and will be discussed below.

Let **A** be any category (we will require **A** to have a few limits and colimits below). If T is a group, a T-*object* in **A** [Tondeur 1965] is a pair (A, π) where A is an object of **A** and π; $T \to \mathrm{Aut}(A)$ is a group homomorphism. The reader may formulate 'equivariant map' so as to obtain the category $\mathbf{A}^T$ of T-objects in **A**, generalizing $\mathbf{Top}^T$.

Generalizing the inclusion map of $\mathbf{Z}$ in $\mathbf{R}$, let $\psi: T \to S$ be a group homomorphism and let $U_\psi: \mathbf{A}^S \to \mathbf{A}^T$ be the obvious induced functor. Let (A, π) be a T-object in **A**. Let $\{\rangle r, s; t\langle\}$ $\{\langle r, s; t\rangle\}$ denote, respectively, that $r, s \in S$, $t \in T$ and $\psi(t) = \{sr^{-1}\}$ $\{rs^{-1}\}$. Let $A_s = A$ for each s in S. Then the free S-object over (A, π) with respect to U_ψ (that is, the value of the left adjoint of U_ψ on (A, π)) is constructed as the colimit of the diagram (shown) of **A**-morphisms with nodes $\{A_s\colon s \in S\}$ and edges $\pi(t)\colon A_r \to A_s$ whenever $\rangle r, s; t\langle$. The cofree object is similarly constructed as the limit of $\pi(t)\colon A_s \to A_r$ with $\langle r, s; t\rangle$. (All of this can be done in the relative case [Dubuc 1970]. This is necessary for obtaining the tensor and hom functors induced by U_ψ the 'change of rings' functor in module theory, and is also necessary in the Kelley space categories studied by de Vries.)

$$\begin{tikzcd} A_r \arrow[rr, "\pi(t)"] \arrow[dr] & & A_s \arrow[dl] \\ & C & \end{tikzcd} \qquad \begin{tikzcd} A_s \arrow[rr, "\pi(t)"] & & A_r \\ & L \arrow[ul] \arrow[ur] & \end{tikzcd}$$

colimit for left adjoint — limit for right adjoint

Let us now turn to the case of interest, $U_\psi\colon \mathbf{Top}^S \to \mathbf{Top}^T$. Strictly speaking, this is an example of the general case discussed above only if S and T are discrete. However the guesswork required to generalize the discrete case is very minor, amounting to 'try continuous maps with the compact-open topology'. The extra assumptions we impose are that ψ is a continuous group homomorphism and that S is locally compact Hausdorff.

As a side observation, notice that U_ψ explains the categories **TTG** and $\mathbf{TTG}_*$. For $(f, \psi)\colon (X, T, \pi) \to (Y, S, \rho)$ in **TTG** if and only if $f\colon (X, \pi) \to U_\psi(Y, \rho)$ in $\mathbf{Top}^T$, and $(g, \psi)\colon (X, S, \pi) \to (Y, T, \rho)$ in $\mathbf{TTG}_*$ if and only if $g\colon U_\psi(X, \pi) \to (Y, \rho)$ in $\mathbf{Top}^T$. Finding adjoints represents these as morphisms in $\mathbf{Top}^S$.

Fix (Y, ρ) in $\mathbf{Top}^T$. A *ψ-motion* in (Y, ρ) is a $\mathbf{Top}^T$-morphism $f\colon U_\psi S \to (Y, \rho)$ (where S acts on itself by group multiplication), that is, f is continuous and $f(s \cdot \psi(t)) = \rho(f(s), t)$. Notice that f is an extension of the T-motion of $f(e)$ as shown in the diagram.

$$\begin{tikzcd} S \arrow[drr, dashed, "f"] & & \\ & & Y \\ T \arrow[uu, "\psi"] \arrow[urr, "\rho(f(e), -)"'] & & \end{tikzcd}$$

Let $\mathrm{Mot}_\psi(Y, \rho)$ be the set of all ψ-motions in (Y, ρ) with the compact-open topology. Then $(f, s) \mapsto fL_s$ is an action of S on $\mathrm{Mot}_\psi(Y, \rho)$ and is a continuous action by de Vries 2.1.3. This construction provides the cofree object over (Y, ρ), the unique coextension $h_\#$ in $\mathbf{Top}^S$ as shown in the diagram being defined by $h_\#(x) = h\pi(x, -)$. In the special case where (Y, ρ) is $U_\psi(X, \pi)$ and $h = \mathrm{id}$, $h_\#$ is called the *costructure map* of (X, π) (the terminology is standard in monad theory).

For an example, let $T = \mathbf{Z}$ and $S = \mathbf{R}$ and consider the flow $(\mathbf{R}^2, \mathbf{R}, \pi)$

$$\begin{tikzcd} \mathrm{Mot}_\psi(Y, \rho) & \\ & (X, \pi) \arrow[ul, dashed, "h_\#"'] \end{tikzcd} \qquad \begin{tikzcd} U_\psi \mathrm{Mot}_\psi(Y, \rho) \arrow[rr, "\mathrm{pr}_e"] & & (Y, \rho) \\ & U_\psi(X, \pi) \arrow[ul, dashed, "U_\psi h_\#"] \arrow[ur, "h"'] & \end{tikzcd}$$

induced by $\ddot{x} = -x$. If $\psi(n) = 2\pi n$, every point of $U_\psi(\mathbf{R}^2, \pi)$ is stationary. $\mathrm{Mot}_\psi U_\psi(\mathbf{R}^2, \pi)$ is the loop space of $\mathbf{R}^2$. On the other hand, if $\psi(n) = 2\pi\alpha n$ with α irrational, the costructure map of $(\mathbf{R}^2, \pi)$ is an isomorphism.

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