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Mathematical cosmology and extragalactic astronomy, by Irving Ezra Segal, Pure and Applied Mathematics, vol. 68, Academic Press, New York, 1976, ix + 204 pp., $22.50.

For more than half a century it has been observed that there exist shifts in frequency of light emitted by distant sources. The redshift parameter z is defined as the fractional *increase* in wave length

$$z \equiv \delta\lambda/\lambda_1 = (\lambda_0 - \lambda_1)/\lambda_1$$

where λ_1 is the wave length of the emitted light and λ_0 is that of the received light. If ν_1 and ν_0 are the corresponding frequencies we may write

$$1 + z = \nu_1/\nu_0.$$

Segal devotes approximately the first half of this book to a discussion of the implications that the requirements of causality and symmetry have on any cosmological theory. In addition this part of the book contains a derivation of the variation of the redshift z as a function of the distance ρ from the point of emission in accordance with the law $z = \tan^2[(\rho/2R)]$ where R is the "radius of the Universe". He deduces from this law the functional dependence of a variety of observed quantities such as the apparent luminosities, number counts and apparent angular diameters on z. The remainder of the book is devoted to comparing these predicted relations with observations. For small z, Segal's redshift distance law differs markedly from the linear law proposed by Hubble, which in turn is in accordance with the expanding universe models predicted by general relativity, i.e. the Einstein theory of gravitation.

In order to be able to point out the relation between the latter theory and chronometric theory, the one propounded by Segal, it is appropriate to summarize some features of the general relativistic treatment of the expanding universe. The Einstein theory states that the arena in terms of which physical theories are to be described is a four-dimensional manifold called space-time with a Lorentzian metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu.$$

The metric tensor $g_{\mu\nu}$ describes the gravitational field and is determined by the field equations

$$R_{\mu\nu} - (R/2)g_{\mu\nu} + \Lambda g_{\mu\nu} = -KT_{\mu\nu}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and scalar curvature computed from $g_{\mu\nu}$. $T_{\mu\nu}$ is the stress-energy tensor describing the sources of the gravitational field, K is the Einstein gravitational constant and Λ is the cosmological constant. Λ was introduced into the theory by Einstein in his first discussions of the cosmological problem. He subsequently felt very strongly that $\Lambda = 0$ and that his introduction of this constant was one of his most serious errors (cf. [**1**]).

When information regarding $g_{\mu\nu}$ is available, the field equations provide us with information concerning $T_{\mu\nu}$, i.e., the sources. The most specialized of general relativistic cosmologies do provide information regarding the $g_{\mu\nu}$ for such theories assumed that space-time admits a family of three-dimensional space-like hypersurfaces labelled by a parameter t, the cosmic time. These hypersurfaces are further assumed to admit a six parameter group of motions. Thus the space part of space-time, the hypersurfaces t = constant, are assumed to be homogeneous and isotropic. It then follows that we may introduce coordinates in terms of which the line element may be written as

$$ds^2 = dt^2 - R^2(t)\,dl^2$$

with

$$dl^2 = \frac{1}{(1 + kr^2/4)^2}(dx^2 + dy^2 + dz^2), \qquad k = 1, 0, -1, r^2 = x^2 + y^2 + z^2.$$

We may also write

$$dl^2 = d\omega^2 + \sigma^2(\omega)(d\theta^2 + \sin^2\theta d\varphi^2)$$

where

$$\sigma(\omega) = \begin{cases} \sin\omega & \text{when } k = +1, \\ \omega & \text{when } k = 0, \\ \sinh\omega & \text{when } k = -1. \end{cases}$$

A space-time with a fixed k and particular function $R(t)$ is said to be a cosmological model or an expanding universe. Each cosmological model is conformally flat, that is we may find coordinates T, X, Y, Z such that

$$ds^2 = F^2(dT^2 - dX^2 - dY^2 - dZ^2)$$

and F is a function of these coordinates. Hence, each of these space-times admits the 15 parameter group of conformal transformations of Minkowski space. Further, each expanding universe may be regarded as a four-dimensional surface in five-dimensional Minkowski space.

There are four cosmological models that are stationary. That is, each one of these admits an additional one parameter group of motions such that the vector tangent to the orbits of the group, the Killing vector generating the group is time-like. These are given by $R(t)$ = constant and $k = -1, 0, +1$, and $k = 0$, $R = e^{t/b}$, b a constant. The latter universe is known as the De Sitter universe. The one with $k = 1$, R = constant is called the Einstein Universe.

When the Einstein field equations are used to compute $T_{\mu\nu}$, the source of the gravitational field, we find that this is the stress energy tensor for a perfect fluid. Each fluid particle is at rest in the coordinate system in which the metric is given as above and the pressure p and energy density ρ are given by

$$Kp = \Lambda - \frac{1}{R^2}(2\ddot{R} + \dot{R}^2 + k), \quad K\rho = \frac{3}{R^2}(\dot{R}^2 + k) - \Lambda$$

in units in which the velocity of light has been set equal to one. The line element given above for such models is called the Robertson-Walker one, and the models for which $R(t)$ is determined via the field equations with $\Lambda = 0$ are often referred to as Friedmann models. When the pressure is set equal to zero, the field equations determine the function $R(t)$ in terms of the two constants

$$H_0 = \left(\frac{\dot{R}}{R}\right)_{t=t_0} \quad \text{and} \quad q_0 = -\frac{1}{H_0^2}\left(\frac{\ddot{R}}{R}\right)_{t=t_0}.$$

The first is known as the Hubble constant and the second as the deceleration parameter.

The behaviour of physical systems such as fields and particles whose own gravitational fields are negligible are described by tensor and/or spinor fields defined in the space-times with a Robertson-Walker metric. In particular, the Maxwell electromagnetic fields describing light are defined by an antisymmetric second order tensor which obeys a set of partial differential equations involving the covariant derivatives of this tensor field.

The history of an observer is represented by a time-like curve in space-time, his world line. His observing apparatus involves a tetrad of vectors defined along this curve and the observations he makes on various tensorial fields are expressed in terms of the components of the tensor relative to this tetrad.

However, the behaviour of light rays may be discussed without solving the Maxwell equations for it may be shown that light travels along null geodesics in space-time. Thus one may derive a redshift distance relation from a Robertson-Walker metric by observing that the distance travelled by a light ray emitted at time t_1 and received at time t_0 is given by

$$\omega = \int_{t_1}^{t_0} \frac{dt}{R(t)}.$$

The light will undergo a redshift given by

$$1 + z = R(t_0)/R(t_1).$$

By using power series expansions we may express ω as a function of z and obtain for small z,

$$\omega = \frac{z}{H_0 R_0}\left[1 - \frac{1}{2}(q_0 + 1)z + \left(1 + q_0 + \frac{q_0^2}{2} - \frac{\beta}{6}\right)z^2 + \cdots\right]$$

where

$$\beta = \frac{1}{H_0^3}\left(\frac{\dddot{R}}{R}\right)_{t=t_0}.$$

This redshift distance relation for small z is a linear one. Such a linear relation was proposed by Hubble. The redshift distance relation and knowledge of the spatial geometry enable one to derive a variety of relations between observable quantities. Hubble assumed that the spatial geometry was Euclidean and interpreted various observations on this basis. Segal refers to derivations involving the assumptions made by Hubble as the "Hubble model (or theory)". The results of derivations based on a Robertson-Walker space-time differ from those of the Hubble model and from those of Segal primarily because the redshift distance relations used differ.

Segal's discussion of the nature of space-time is similar in many ways to that occurring in general relativity. He differs markedly from that theory in his definitions of observer, clock and rod. Both theories are chronogeometric, that is, in them "considerations of temporal order are merged with geometry in a mathematical way". In addition, both theories use as a starting point for causality considerations a structure involving the assigning of a closed convex cone in the tangent space at each point of the manifold called space-time.

In Segal's theory group invariance properties are assumed and exploited. Thus the causal manifold is assumed to admit a nontrivial class of "temporal displacements" which are automorphisms of the manifold (qua causal manifold). A clock is defined in terms of the parameter of a one parameter group of such temporal displacements. The clocks considered in relativity theory, defined by means of the arc-length of a time-like world-line in the manifold differ from the clocks defined above when the space-time in Minkowski space and the world-line is not a straight time-like line or when the space-time has a nonflat metric with no time-like Killing vector field.

The book contains a meticulous and concise description of the interplay between the ideas involved in causal manifolds admitting groups that preserve their structure. Causality in groups and causal morphisms of groups are also treated. The mathematical discussion of these topics lays a foundation for Segal's derivation of the redshift distance relation. This is quite different from that outlined above for the general relativisitc theory. He makes a series of assumptions regarding the nature of space-time which restrict the space-times to be one of three Lorentzian manifolds. Segal selects one of these as the physical space-time (denoted by him as the Cosmos). It is the universal covering manifold $\tilde{M}$ of the conformal compactification of Minkowski space. The other two possibilities are simply derivable from it. Minkowski space osculates the physical (universal) space-time in that suitably scaled Minkowski and universal space-time agree to terms of second order in the reciprocal of the "radius of the universe". $\tilde{M}$ is the space-time previously described as the Einstein Universe. The assumptions made by Segal are similar to those made in deriving the Robertson-Walker metrics, as is to be expected from his notion of observers and clocks. He makes the additional assumption that space-time is stationary.

Segal states that "observed fields and particles are appropriately described by functions defined on the Cosmos with values in a suitable spin-space, . . . ". Presumably these functions are determined by partial differential equations which are hyperbolic in a suitably defined sense. Such functions describe the dynamical behavior of physical systems. No equations of this sort are presented or discussed in this book. Instead of proceeding as is done in special and general relativity and relating physical measurements to scalars derived from tensor or spinor fields defined over space-time (the Cosmos), Segal postulates "that anthropomorphically possible local measurements are represented theoretically by the *flat* rather than *curved* dynamical variables; while on the other hand the 'true' nonanthropomorphic dynamics and analysis is curved (in the fasion appropriate to the unispace cosmos) rather than flat. That is, we measure the flat dynamical variables; but the Universe in the large runs on the curved basis ...".

The flat dynamical variables referred to in the above quotation are functions of coordinates in a Minkowski space constructed at each point of unispace (the Cosmos) as follows: If t, r, θ, φ are inertial polar coordinates in Minkowski space, the metric there is given by

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2.$$

Let

$$\frac{r+t}{2R} = \tan\left(\frac{\tau+\rho}{2R}\right), \quad \frac{r-t}{2R} = \tan\left(\frac{\rho-\tau}{2R}\right).$$

It then follows that

$$ds^2 = \frac{R^2}{\cos^2((\tau'+\rho')/2)\cos^2((\rho'-\tau')/2)}[(d\tau')^2 - (d\rho')^2 - \sin^2\rho' d\Omega^2]$$

with $\tau' = \tau/R$ and $\rho' = \rho/R$. Variables t, x, y, z obtained from t, r, θ, φ by the usual polar coordinate transformations are said to be "local Minkowski coordinates" associated with the origin in unispace. Any other event in unispace, say the one with coordinates τ'', ρ'', θ, φ, will have local Minkowski coordinates obtained by formulas given above with ρ and τ replaced by $\rho - \rho''$ and $\tau - \tau''$, respectively.

The flat dynamical variables that Segal assumes to represent measurements are functions of the Minkowski coordinates, whereas the true dynamical ones are functions of the variables in unispace.

Segal assumes that the source of the light, received and measured by an observer using local Minkowski coordinates is at rest in unispace. The observer ascribes a redshift to the light emitted by the source because of his measurement procedures based on local Minkowski coordinates. As mentioned above this redshift is computed to be

$$z = \tan^2(\rho/2R)$$

where ρ is the unispace distance between the source and observer and R is "the radius of the Universe". The redshift is thus said to be due to a pseudo-kinematical effect arising from measurement techniques based on local Minkowski spaces.

The chronometric theory as described in this book is not a theory concerning the nature of the universe nor the behaviour of objects in it. Rather it ignores the effect of gravitational forces on these objects, postulates that astronomical bodies in it are at rest without explaining how this happens and ascribes the redshift to a particular description of methods of measurement which is at variance with that used in theories such as general relativity.

The role that the conformal group displays in unispace is also displayed in each Robertson-Walker space-time since they are all conformally flat; but not all of these metrics are stationary. However, Segal is prepared to allow the scale quantity R occurring in the equations given above to be a function of position and in particular of time in unispace. In the latter case unispace could be represented by a space-time with a Robertson-Walker metric. Chronometric theory would then differ from general relativistic cosmology in that it does not have any field equations from which to determine the matter content of the universe and the dynamical behaviour of this matter; nor does it use the same methods for relating functions defined on the space-time to physical measurements.

Segal devotes the last half of his book to a discussion of the relation between observations and predictions based on the square-law redshift-distance dependence for sufficiently small distances given by chronometric theory. An extensive amount of astronomical data is surveyed and analyzed. Segal claims that he finds "confirmation for the square-law in a number of observational studies at moderate redshifts and overwhelming evidence for a phenomenological square-law in the case of low-redshift galaxies".

This view of the observational data is not universally agreed to. Many astronomers and relativists feel that Sandage's painstaking work **[2]**–**[8]** on determining the apparent magnitude-redshift relation for the brightest galaxy in each of 98 clusters of galaxies is more in accord with a linear redshift-distance dependence than a square one for sufficiently small distances. Segal comments at length on some of this work but discounts it in the main saying that "while undoubtedly of outstanding accuracy (it) appears to be of uncertain statistical uniformity".

Recently Kollerstrom and McVittie **[9]** have analyzed Sandage and Hardy's data given in **[8]**. There the corrected apparent magnitude V of the brightest galaxy in 98 clusters of galaxies is given as well as the redshift corresponding to V.

In **[9]** the quantity $Y = V - 5 \log z$ was studied as to its dependence on a power series in z, z^{-1} and as one in $\log z$. It was concluded that "there is no significant dependence of Y on any moderately smooth function of z" (other than a constant). The chronometric theory would predict that $\partial V/\partial \log z$ is 2.5 instead of 5.

Throughout the second part of the book where observational data are discussed and compared to theoretical predictions, the phrases "Hubble theory", "expansion hypotheses", "general relativistic models", and "Friedmann model" are used. The first of these is described briefly in a footnote. The others are never explicitly defined. As a result this reader was in some doubt as to how to interpet various comments made in this portion of the book. Segal makes claims that "the chronometric theory provides a much better overall fit to extragalactic data than do straightforward general relativistic models with free parameters q_0 and Λ". However, very little of the observational data is compared to the predictions based on ranges of values of q_0 and Λ. Incidentally, these quantities are never defined in this book. Neither is the

Hubble constant (parameter), nor is there any indication as to how the chronometric theory enables one to study the variation of this quantity with distance. Thus it is difficult to evaluate the claim (cf. p. 118) "a further advantage of the chronometric theory over the expansion-theoretic model is that it reconciles the different values (of the Hubble constant) on the basis of different distances to the objects under observation".

I found this a difficult book to read in part because various definitions and derivations were omitted. Nevertheless, I consider that the comparison made above between chronometric theory and general relativistic cosmology an accurate one. I do not agree with comments made by Segal about general relativity and its degree of experimental verification.

This book has not convinced me that chronometric theory is a replacement for general relativistic cosmology, a branch of a theory which contains Newton's theory of gravitation as a limiting case and which provides observed corrections to that theory.

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Cohomology theory of topological transformation groups, by Wu Yi Hsiang, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 85, Springer-Verlag, New York, Heidelberg, Berlin, 1975, x + 164 pp., $25.00.

The theory of finite (and generally compact) groups of transformations of manifolds had its origins slightly over half a century ago in the work of Kerékjárto [34] and Brouwer [12] showing that periodic transformations of the 2-disk and 2-sphere are topologically equivalent to rotations. (An error in the original proof was later corrected by Eilenberg [20].) Similar results for actions of compact connected groups on 3-space were proved by Montgomery and