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Sheaf theory, by B. R. Tennison, London Mathematical Society Lecture Note Series, No. 20, Cambridge University Press, New York and London, 1976, vii + 164 pp., $8.95.

Perhaps the most pressing problem facing mathematics today is the increasing difficulty in communicating with nonmathematicians. The low percentage of new math Ph.D.'s with nonacademic jobs, the almost nonexistent intellectual interaction with other academic departments, and the increasingly common practise of having nonmathematicians teaching mathematics in their own disciplines illustrate this problem. In large measure it has been caused by an unhealthy overemphasis on abstraction during the past few decades. This particular book and, for that matter, all of the other books devoted solely to sheaf theory are prime examples of this overemphasis.

Since the mathematical style of graduate level texts is an important factor in determining the tastes of new mathematicians, these books and others which are written without reference to the concrete problems that gave rise to modern day mathematical edifices endanger the development of mathematics. The mathematical standards that are developed in our graduate students demand abstraction and elegant generalization while doing away with the necessity of justifying a result in terms of potential applications. This is a natural consequence of courses that rarely, if ever, present as a central topic a mathematical question of interest to a physicist or economist and then answer it in terms they could hope to understand. Instead a great deal of unnecessary generalization is introduced and stressed. As a result of this training in generalization, our new Ph.D.'s know how to check a theorem by determining its logical consequences or varying its hypotheses, but they rarely know how to apply the theorem to a problem of interest to a nonmathematician.

In the hands of an expert the power of abstraction and generalization is clear as Deligne's recent proof of the Ramanujan conjecture shows. Deligne was able to reduce this concrete conjecture about the partition function to the characteristic p Riemann hypothesis, and then by using the abstract, 'general nonsense' machinery of Grothendieck topologies and Grothendieck sheaf theory, he was able to prove the latter conjecture by an ingenious argument. Unfortunately in the hands of a novice mathematician, the power of abstraction and generalization too often leads to new "results" in abstract areas such as category theory, point set topology, or universal algebra while also giving him the impression of having done real mathematics. We badly need to correct this impression by emphasizing that the quality of a result is in large part determined by what it says about basic physical and mathematical problems.

Unfortunately the book under review will not, indeed, cannot, do this. It is a book devoted to a language, the language of sheaves, which may, by the end, leave the inexperienced reader with the feeling that he has been introduced to real mathematics. It is written in a crisp, clear, generally appealing style. It can be read easily by graduate students since nothing deeper than the basic properties of sheaves, Čech cohomology, etc. is discussed. It provides an example to graduate students of how category theory may be 'applied' to mathematics. In short it is a seductive introduction to fashionable mathematics. And yet there are no indications of why a nonmathematician should be interested in sheaf theory (if in fact he should be), and there are only a few hints of what its importance is to a mathematician (if in fact there is any–after all, geometers survived for many years with "maximal covering atlases" !). A better sense of perspective would surely have been maintained if Tennison had emphasized that sheaf theory was a useful language with no intrinsic value, instead of implying otherwise by stating in the introduction: "The approach to the subject taken here is rather categorical, and the course may be used ... as an introduction to the usefulness of categories and functors."

Nevertheless, it is a language of utility and wide use in certain areas of mathematics, particularly algebraic geometry. Consequently, this book could be a useful adjunct to an abstract second course in algebraic geometry or to a course in Grothendieck topologies. The first three chapters provide the basic definitions and properties of presheaves, sheaves, and sheaf morphisms. The fourth chapter deals with ringed spaces and develops sketchily the prime spectrum of a ring. The last chapter defines sheaf cohomology via both injective resolutions and the Čech approach. Category language is used throughout although the sheafification functor is introduced via sheaf spaces and the words "left adjoint" do not appear until p. 61 (before that, the Hom isomorphism is written out).

The selection of material from sheaf theory is good. It wisely avoids attempting to cover all of the material in Godement or Bredon, but by the end the reader who is at ease with the basic definitions of category theory as given, say, in Freyd, *Abelian categories*, will be able to use the language of sheaf theory in most situations and will also have absorbed the basics of homological algebra. The exercises are good and even include some which lead into logical topoi. Examples are not emphasized in the text. There is a section on the prime spectrum of a ring which is much too skimpy but refers to two other sources for more details. It is followed by a section defining manifolds–topological, differentiable, complex analytic, etc. The last major example is the picard (sic) group of a ringed space X. It is defined and the isomorphisms

$$\mathrm{Pic}(X) \cong \check{H}^1(X, O_X^*) \cong H^1(X, O_X^*)$$

are proved. But then legitimate applications of sheaf theory are hard to find since it is only a language.

From the standpoint of Grothendieck topologies, the wrong definition of the sheafification functor is given. This is surprising, given Tennison's consistently categorical orientation. The sheafification functor L is introduced by passing from a presheaf to the associated sheaf space and then via continuous cross sections to the associated sheaf. Since Exercise 4.10 in the last chapter develops the + construction and shows that $F^{++} = L(F)$, a judicious rearrangement of the material introduces the sheafification functor so that it can be immediately extended to general Grothendieck topologies as in Artin's marvelous introduction to the subject [1]. There is also a surprising ambiguity in his attitude towards adjoint functors. While category language is emphasized and used throughout, left adjoint is not mentioned until well into the material. The new student to category theory will appreciate always having the Hom isomorphism written out, but those familiar with the term will wonder why no use is made of it in proving exactness properties or why no mention is made of the sheafification functor being a left adjoint except in an exercise fifteen pages after its definition.

There are several minor failings in this book. While most of the notation is standard, C^Y is used for the set of continuous functions from X to Y. In Exercise 5.7, "fractional" must be replaced with "invertible". In the last chapter, the definition of "effaceable" on p. 128 is too restrictive, and this makes the skeletal argument at the top of p. 139 confusing to the congnescenti while being much too sketchy for the novice. The latter is a criticism that can be made in several other places also. A reference to Hartshorne, *Local cohomology*, should be added to those given on p. 140 since its approach to sheaf supports is much closer to Tennison's style than Swan or Bredon.

But in terms of what Tennison tried to do, these are minor flaws that are easily corrected, and so the book could serve as a useful supplemental text in a graduate course using sheaf theory. It would have been better, however, if the book had never been written. After all, the basic definitions and properties of sheaves are not very difficult to grasp. Sheaf theory should be a chapter in a book on several complex variables or algebraic geometry or differential geometry or .... With the applications immediately at hand, it is much easier to maintain a proper perspective.

BIBLIOGRAPHY

1. M. Artin, *Grothendieck topologies*, Mimeographed notes, Harvard University, 1962.

Probably the best introduction to Grothendieck topologies around, at least, until Deligne's appears, except that the arrows between pretopologies go the wrong way. These notes should be reprinted somewhere.

2. G. E. Bredon, *Sheaf theory*, McGraw-Hill, New York, 1967. MR **36** #4552.

The American version of 4.

3. P. Freyd, *Abelian categories*, Harper & Row, New York, 1964. MR **29** #3517.

A brief, elegant, insidious introduction to category theory presenting the same problems that Tennison's book does. Also the source of that terrible pun: A small category is a kittygory.

4. R. Godement, *Topologie algébrique et théorie des faisceaux*, Actualités Sci. Indust. no. 1252, Hermann, Paris, 1958. MR **21** #1583.

Everything is here, but not in so categorical a style.

5. R. Hartshorne, *Local cohomology*, Lecture Notes in Math., vol. 41, Springer-Verlag, Berlin and New York, 1967. MR **37** #219.

Grothendieck's categorical approach to sheaf cohomology with supports. It contains the seeds of Poincaré duality.

6. C. Linderholm, *Mathematics made difficult*, World Publishing, New York, 1972.

For those interested in seeing what can be done with category theory.

R. T. HOOBLER