In the sixties I tried without success to find such a theory, or to imbed the Morse-Tompkins-Shiffman result in a conceptual general setting. Tromba and Uhlenbeck may now have succeeded in initiating a development of calculus of variations in the large for more than one independent variable.

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Rings of quotients, by Bo Stenström, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 217, Springer-Verlag, New York, Heidelberg, Berlin, 1975, viii + 309 pp., $39.60.

Noncommutative ring theorists have long been tantalized by the method of localization used so easily and successfully by their commutative colleagues. It is unfortunate, yet typical, that results and techniques which are almost trivial for commutative rings turn out to be either false or impossible for noncommutative rings. Stenström's *Rings of quotients* records the attempts at developing a comprehensive, general technique of localization for noncommutative rings.

The study of quotient rings for noncommutative rings goes back to the early 1930s with the question in van der Waerden's first edition about whether noncommutative integral domains could be embedded in division rings. Ore, in 1931, found a criterion (the "Ore condition") for an integral domain to have a division ring of fractions: Given nonzero elements a and b, there exist nonzero c and d such that $ac = bd$. Independently, Wedderburn, in 1932, proved directly, by a similar procedure, that Euclidean domains have division rings of fractions.

The subject attracted little interest until the early fifties. There was, however, an important development due to Asano [**1**]. Asano's result was of less interest than his method, both of which will be described here. If R is a commutative ring and S a multiplicatively closed subset of non-zero-divisors, then it is an easy matter to construct a ring, $R[S^{-1}]$, containing R and with the properties:

(1) If $s \in S$, then s is invertible in $R[S^{-1}]$,

(2) If $x \in R[S^{-1}]$, then $x = rs^{-1}$, $r \in R$, $s \in S$.

Generalizing Ore's theorem, Asano showed that for arbitrary rings R, and multiplicatively closed subsets of non-zero-divisors S, such $R[S^{-1}]$ exist if and only if, given $r \in R$ and $s \in S$, there exist $r' \in R$ and $s' \in S$ such that $rs' = sr'$. When S consists of all non-zero-divisors, $R[S^{-1}]$ is the (right) classical ring of quotients of R–a case already treated by Jacobson in 1943 [**4**].

Asano's method was to consider a "fraction" as a partial endomorphism of the ring R. To wit: Let F be the family of all right ideals containing an element of S. The Ore condition ensures that if $I_1, I_2 \in F$ then $I_1 \cap I_2$ is in F, and if $a \in R$ and $I \in F$ then $a^{-1}(I) \in F$. Asano now considers the union of the groups $\mathrm{Hom}_R(I, R)$ for $I \in F$. On this union we can define an equivalence relation: $f_1 \sim f_2$ if f_1 and f_2 agree on the intersection of their domains. These equivalence classes can be made into a ring with the desired properties. As Lambek has pointed out: 2/3 may be viewed as the map from $3Z$ to Z which sends $3n$ to $2n$.

Although there was scant progress for noncommutative rings, the subject achieved its modern form for commutative rings in the work of Uzkov [**10**] who showed how to localize at an arbitrary multiplicatively closed subset not containing 0.

Johnson [**6**], building on Asano's work, and Utumi [**9**], on Johnson's, constructed rings of "quotients" which, while not rings of "fractions", possessed some of the latter's more important properties. These new constructions replace Asano's family of right ideals (which is, infrequently, well behaved) with other families. Johnson, for example, used the essential right ideals–those which have nontrivial intersection with every proper right ideal.

This is a good place to pause and consider just what should be expected from a "ring of quotients". There are two desiderata: (1) There should be a "tight" connection between the ring and the quotient ring, and (2), the ideal structure of the quotient ring should, in some sense, be better than or easier to analyze than that of the original. Surely, the ring should be no worse. Unfortunately, often the more general rings of quotients turn out to be less useful than one would hope because their ideal structure is often harder to study than that of the original ring. In fact, the entire subject of quotient rings might be but a remote backwater visited only by lost Ph.D. students if it were not for an unexpected and revolutionary result–Goldie's theorem.

This theorem does for rings with maximum conditions what the Wedderburn theorems do for rings with minimum condition and, indeed, forms a link between the two theories. Goldie ([**2**] and [**3**]) characterized all rings possessing a classical ring of quotients which is semisimple Artinian. In particular, he showed that a right Noetherian ring with no proper nilpotent ideals has such a quotient ring.

Goldie's papers are equally important for the methods they introduce; for example, the careful study of annihilators. It is Goldie's work which breathed new life into much of noncommutative ring theory and touched all of it somehow. Withal his techniques did not, even for Noetherian rings, yield the kind of localization technique available in the commutative theory–and for good reason. The theorems that one would hope to prove for such rings were simply not true!

Various techniques designed to emulate commutative localization at a prime have been devised by Goldie, Lambek and Michler among others. They have been far from satisfactory–a localization of a Noetherian ring need not even be Noetherian, for instance.

Stenström's volume is a comprehensive and clear, albeit stodgy, exposition of these matters and more. The book also gives a useful account of homological and categorial methods in noncommutative ring theory. The author, for the most part, relegates the examples to the copious exercises, history to the Introduction, and motivation, apparently, to the reader. It seems like such a little thing to say why one is doing thus and so and not perhaps something else. Ah, well ....

Although the book is fairly complete there are the inevitable omissions, some of which the author acknowledges: Posner's theorem, the Faith-Utumi theorem, etc. Also, it would have been nice to give an account of Martindale's central closure [7] as an application of the more general constructions to concrete problems. Too late for inclusion, the reviewer supposes, is the principal ideal theorem of Jategaonkar [5], the proof of which is inspired by the torsion theoretic methods on which the book spends so much time.

The book is remarkably free of errors (exercise 8 on p. 61 is false as an example of Bergman shows) and the bibliography is excellent. All things considered the reviewer feels that *Rings of quotients* will be useful to advanced students as a text and to workers in the field as a reference.

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