domain element, whereas a change in a single Walsh coefficient is felt throughout the domain.

At this point in time the work appears to be highly developed and rich in mathematical elegance. It is not clear what the long term directions of the research are, nor what the present implications are. Since the research is largely stimulated by the need to solve practical problems in computer design, one might measure the impact of the research on present design. The impact, unfortunately, has been quite small, and is not likely to improve over time. The cost functions on which the research is predicated have turned out largely to be unrealistic characterizations of present technology, although they were reasonable characterizations of past technology. Practitioners today are able to use canonical realizations with or without small improvements from *ad hoc* analysis to design computers, and the costs of nonminimal circuits have been very close to the costs of absolutely minimal circuits. The theory no longer has to satisfy past constraints and may be driven in new innovative directions.

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Global variational analysis: *Weierstrass integrals on a Riemannian manifold*, by Marston Morse, Mathematical Notes, Princeton University Press, Princeton, New Jersey, 1976, ix + 255 pp., $6.50.

The first thing that comes to mind in reviewing a new book by Marston Morse on the calculus of variations is that he wrote a book, *The calculus of variations in the large*, forty years ago. The early book gave the foundations of what is now called Morse theory. The publication of a new book by Morse on the same subject presents an occasion to give some personal perspectives on how this mathematics has developed in the last few decades. I say "personal perspectives" and indeed, I, myself, have been involved in, and inspired by, Morse's mathematics. For example, three of my papers contain the word Morse in the title. Another mathematician much influenced by Morse, Raoul Bott, was my adviser, and even work of Morse (but not variational theory) suggested to Bott the thesis problem he gave me (leading eventually to my work in immersion theory).

Another factor in writing a review like this is that, today, global analysis is very much alive, both in mathematics and other disciplines. It may give us some perspective to trace the development of one of the main roots of the subject.

Let us see what Morse, in 1934, had to say about global analysis (he used the word macro-analysis, then). I quote the full first paragraph of the Foreword of his book.

"For several years the research of the writer has been oriented by a conception of what might be termed macro-analysis. It seems probable to the author that many of the objectively important problems in mathematical physics, geometry, and analysis cannot be solved without radical additions to the methods of what is now strictly regarded as pure analysis. Any problem which is nonlinear in character, which involves more than one coordinate system or more than one variable, or whose structure is initially defined in the large, is likely to require considerations of topology and group theory in order to arrive at its meaning and its solution. In the solution of such problems classical analysis will frequently appear as an instrument in the small, integrated over the whole problem with the aid of group theory or topology. Such conceptions are not due to the author. It will be sufficient to say that Henri Poincaré was among the first to have a conscious theory of macro-analysis, and of all mathematicians was doubtless the one who most effectively put such a theory into practice."

Note Morse's acknowledgment of the role of Poincaré in this subject. Already in 1885, Poincaré knew of Morse inequalities for a surface.

Note also how Morse feels that problems (of analysis) nonlinear in character are likely to require considerations of topology and group theory. I would like to echo this point, which even today, forty years later, has still not been digested by some analysts, analysts, for example, who are not willing to relinquish their linear spaces and linear space methods to confront nonlinear problems.

Suspicions of geometry and the uses of geometry in analysis have indeed deep roots. Even G. D. Birkhoff wrote in 1938, in *Fifty years of American mathematics*, of his "... disturbing secret fear that geometry may ultimately turn out to be no more than the glittering intuitional trappings of analysis." He used the word geometry to include topology or "analysis situs" as it was called then.

The simplest case of Morse theory is just the phenomenon that a differentiable function on an interval with two local minima must have a local maximum between the local minima. This "minimax principle", while very old, received a big push by G. D. Birkhoff in 1917. It is interesting to see what Morse had to say about some of the origins of the global calculus of variations. In his obituary of Birkhoff, he wrote (on the minimax principle):

"In Birkhoff's applications this principle reduces to an existence theorem for critical points of an analytic function $F(x)$ of n-variables. If one supposes for the sake of definiteness that F is defined over a regular, compact, analytic manifold, then, suitably counted, there exist at least $R_1 + M_0 - 1$ generalized saddle points, where R_1 is the linear connectivity of the manifold and M_0 the number of points (supposed isolated) of relative minima of F. In similar or related forms this principle was known and applied by Poincaré, Maxwell, and Kronecker, and has an origin even more remote in the past. Birkhoff's bold step was to conceive of its application to functions of curves such as the integral J. He applied it in the billiard ball problem **[26]** (motion on a convex table) and to obtain closed geodesics on a convex surface."

Thus, one has here a relation between the analysis, saddle points or geodesics, and the topology, "linear connectivities", or more generally, Betti numbers.

Morse's central contribution was to take these ideas and apply them systematically to a functional on the "manifold" of curves joining two points to deduce the existence of extremals, especially geodesics. In doing so, he developed relations between the critical points and the topology for any (smooth, nondegenerate) function on a compact manifold. In particular, it was a great accomplishment of Morse, in the years 1925–1930, to have given a global geometric abstract base for the variational calculus. To that base we proceed.

A fundamental insight here deals with the problem: How does the topology change as a function passes a nondegenerate critical point? To explicate matters, let $J\colon \Omega \to R$ be a C^∞ real valued function defined on a compact manifold, Ω. We will suppose that all maps are C^∞ and use the symbol Ω for a manifold for reasons that will become more natural later. A point x in Ω is called a *critical point* of J if it has zero derivative, i.e., if $DJ(x) = 0$. In that case the second derivative $D^2J(x)$ is an invariantly defined bilinear symmetric form H_x on the tangent space T_x of vectors tangent to Ω at x. This form is called the *Hessian* and plays an important role in Morse theory.

Recall that for any bilinear symmetric form H on a linear space E, the *index* is the maximum dimension of a subspace on which H is negative definite. The *nullity* of H is the dimension of the *null* space, i.e., the set of v in E such that $H(u,v) = 0$ for all u in E. Then H is *nondegenerate* if its nullity is zero. If H is nondegenerate and E is R^n, then there are linear coordinates u on R^n such that

$$H(u,u) = -\sum_{i=1}^{k} u_i^2 + \sum_{i=k+1}^{n} u_i^2.$$

Here k is the index.

All of these definitions pass over to a critical point. Thus the *index* of a critical point is the index of its Hessian; the critical point is called *nondegenerate* if its Hessian is nondegenerate, etc. A nondegenerate critical point is necessarily isolated. One knows more.

The *Morse lemma* asserts that if a critical point x of a function J is nondegnerate, then there are coordinates u near x to make J quadratic, i.e., so that $J(u) = H_x(u,u)$. My own belief is that the Morse lemma, while nice to know, is not vital, and in fact Morse theory develops more naturally, more conceptually, without it. To understand the topology of a function on a manifold, one uses a Riemannian metric to define gradient lines of the function. The choice of coordinates in the Morse lemma will not respect this metric. On the other hand, simply Taylor's formula already gives sufficient local information to do the "handle attaching".

Perhaps the preceding remark will become clearer as we proceed with our story of what happens to the topology as a nondegenerate critical point is passed. For any real number a let J_a be the set of all x in Ω with $J(x) \leqslant a$. If there is no critical value in the interval $[b,c]$, then $J^{-1}[b,c]$ is differentiably isomorphic to a product, $J^{-1}(b) \times [b, c]$. (A critical value is the value of a critical point.)

Now suppose there is exactly one critical point x^* with value c, and that x^* is nondegenerate with index k. How is the topology of $J_{c+\varepsilon}$ related to that of $J_{c-\varepsilon}$ for small enough ε? The fundamental result is that $J_{c+\varepsilon}$ is $J_{c-\varepsilon}$ together with a cell (or "handle") of dimension k attached. This handle attaching statement has three versions, which relate to three periods in the development of Morse's critical point theory; and the last two relate to the application of this theory to problems of topology.

The first of these versions is on the homology level and states the relative homology (over the rationals) result:

$$H_i(J_{c+\varepsilon}, J_{c-\varepsilon}) = 0 \quad \text{if } i \neq k,$$
$$\dim H_k(J_{c+\varepsilon}, J_{c-\varepsilon}) = 1.$$

Recall k is the index of the critical point.

This result is central in Morse's 1934 book and is used to deduce the existence of critical points. These critical points correspond to solutions of variational problems, in particular, to geodesics as we shall see later. Thus it is used to make a passage from topology to analysis and geometry.

The second version is a sharpening, explicated by Bott in 1959, which puts the theorem on a homotopy level. This result was used by Bott to study the homotopy of Lie groups. In particular, he obtained the first proof of the Bott periodicity theorems this way. Specifically, the stable homotopy groups of the unitary group U and the orthogonal group O satisfy

$$\pi_i(U) = \pi_{i+2}(U), \qquad \pi_i(O) = \pi_{i+8}(O) \quad \text{all } i.$$

Here for example one may think of U as the union of $U(n)$ over $n = 1, 2, 3, \ldots$ and then π_i is the ordinary homotopy group. This result was the starting point of K-theory.

Bott's version of handle attaching is the statement that $J_{c+\varepsilon}$ is homotopically equivalent to $J_{c-\varepsilon}$ with a cell D^k of dimension k attached by a homeomorphism from the boundary ∂D^k of the cell into the level surface $J^{-1}(c - \varepsilon)$. The homotopy statement yields the homology statement as a corollary.

The third version of handle attaching is on the diffeomorphism level which I believe I was the first to explicate. In fact, this was one very important ingredient in my solution of the "higher dimensional Poincaré conjecture", work on "handle body theory", and the structure of manifolds.

To work on the level of differentiable isomorphism, one must thicken D^k to bring the dimension up to the dimension of the manifold, say n. Thus, let a k-*handle* be $D^{n-k} \times D^k$. Then the strongest version of handle attaching asserts that $J_{c+\varepsilon}$ is diffeomorphic to $J_{c-\varepsilon}$ with $D^{n-k} \times D^k$ attached by an imbedding of $D^{n-k} \times \partial D^k$ into $J^{-1}(c - \varepsilon)$. The attaching process involves a smoothing at the corners. This statement yields the homotopy version as a corollary.

For my favorite proof of these theorems, one supposes that the manifold has a Riemannian metric (by imposition, if necessary) and uses the flow defined by the negative of the gradient of J. At the critical point x^*, the linearized flow has two invariant subspaces in the tangent space T_{x^*}. One of these is contracting under the flow, say E^{n-k}, and one is expanding, say E^k (with the dimension of E^k equal to k, the index of x^*). Use these spaces to define a local product structure in the manifold near x^*; then take small disks D^{n-k}, D^k about the origins in E^{n-k}, E^k, respectively. Using Taylor's formula to expand J about x^*, one can show that the flow on the boundary of $D^{n-k} \times D^k$ has the requisite properties to give the attaching statements. The same proof works at the homotopy and diffeomorphism levels, but requires slightly more checking in the latter case.

The passage from the homology version of handle attaching to the Morse inequalities proceeds most simply via the exact homology sequence of a pair.

Suppose that there are real numbers, $c_0 < c_1 < c_2 < \cdots < c_m$, so that each interval (c_j, c_{j+1}) contains the value of exactly one critical point of $J: \Omega \to R$ and all the critical values are in such intervals. For each j, one has the exact homology sequence of vector spaces over the rational numbers, writing J_j for J_{c_j},

$$\to H_i(J_{j-1}) \to H_i(J_j) \to H_i(J_j, J_{j-1}) \to H_{i-1}(J_{j-1}) \to .$$

From linear algebra, summing from $i = 0$ to k yields for each k:

$$\sum_{i=0}^{k} (-1)^{k-i} \dim H_i(J_{j-1}) - \sum_{i=0}^{k} (-1)^{k-i} \dim H_i(J_j) + \sum_{i=0}^{k} (-1)^{k-i} \dim H_i(J_j, J_{j-1}) \geqslant 0.$$

Summing over j and evaluating by the handle attaching theorem gives

$$-\sum_{i=0}^{k} (-1)^{k-i} \dim H_i(J_m) + \sum_{i=0}^{k} (-1)^{k-i} M_i \geqslant 0$$

where M_i is the *Morse type number* or the number of critical points of index i. Let B_i be the ith Betti number, or $\dim H_i(\Omega)$, and since $J_m = \Omega$, we have

Morse inequalities. $\sum_{i=0}^{k} (-1)^{k-i} M_i \geqslant \sum_{i=0}^{k} (-1)^{k-i} B_i$, each $k = 0, 1, 2, \ldots$.

By adding these inequalities for k and $k - 1$, we get

Simple Morse inequalities. $M_k \geqslant B_k$, $k = 0, 1, 2, \ldots$.

In the preceding discussion we have assumed that different critical points took different values; a minor extension in the handle attaching argument can remove that hypothesis. We have also assumed that all of the critical points of J were nondegenerate. Such functions are called *Morse functions.*

How general are Morse functions? Morse has in his 1934 book a prototype of the theorem that Morse functions form an open and dense set among all C^∞ functions. This result is now centrally imbedded in transversality theory and the theory of singularities of maps à la Whitney, Thom and Mather.

Sard's theorem that with enough differentiability the values of critical points form a set of measure zero is basic in this development and it is no accident that Sard was a student of Morse.

It is important to remark that a complement to Morse's work was developed, already by 1930, by Lusternik and Schnirelmann in the Soviet Union. These mathematicians gave a global existence theory for critical points without making use of nondegeneracy hypotheses.

What we have described above is still in the finite dimensional and abstract realm, thus twice removed from Morse's contributions to the study of geodesics. But before we turn to that study, we give a couple of ways this finite dimensional theory has made itself felt in analysis in the domain of our own experience.

One way is in dynamical systems, where starting from a Morse function J on a Riemannian manifold, one obtains a differential equation with a rather simple structure. The negative of the gradient vector field of J has the property that along solutions (of $dx/dt = -\operatorname{grad} J(x)$), J is never increasing. Thus the dynamical behavior permits no nontrivial periodicity or recurrence. Furthermore, the zeroes of this differential equation, coming from a nondegenerate critical point, possess a certain local robust character. If one adds a second condition of transversality, that the asymptotic sets (the "stable and unstable manifolds") of these zeroes intersect transversally, one can obtain such properties globally. Via this route is the result I obtained with Jacob Palis, that every compact manifold supports a structurally stable dynamical system (so that the "phase portrait" or qualitative structure persists under perturbations). Moreover, it was the interplay between dynamical problems and topology that helped lead me to the handlebody results mentioned earlier.

In this vein, it is worth remarking that the catastrophes of Thom deal with bifurcations of gradient dynamical systems.

A second example is in celestial mechanics where one can show that the relative equilibria in the Newtonian n-body problem correspond to critical points of the Newtonian potential function on complex projective space, properly interpreted. Morse theory of this function suggested to Julian Palmore the existence of new relative equilibria in the 4-body problem which he found.

Let us turn now to the variational theory of Morse, the ultimate object of the abstract theory previously discussed and the subject proper of the book under review. The prinicpal example is the global study of geodesics on a manifold. Let us see how this goes.

Let M be a Riemannian manifold. Recall that this equips each tangent space $T_x = T_x(M)$, x in M, with a norm written $\| \ \|_x$ or sometimes simply $\| \ \|$, defined by an inner product $(\ , \ )_x$ in T_x. One can define the length $l(\alpha)$ of a curve α: $[a, b] \to M$ by

$$l(\alpha) = \int_a^b \|\dot{\alpha}(t)\| \, dt \quad \text{where } \dot{\alpha} = \frac{d\alpha}{dt}.$$

From this, define a metric on M by letting $d(p, q)$ equal the infimum of $l(\alpha)$ over all curves α joining p to q. This is indeed a metric. We will assume that M is complete for this metric.

On M, let points P, Q be given. Denote by $\Omega = \Omega_{P,Q}(M)$ the *loop space*, or set of all (C^∞) curves on M from P to Q. That is,

$$\Omega = \{\alpha \colon [0, 1] \to M | \alpha(0) = P, \alpha(1) = Q\}.$$

The *energy* is the map $J \colon \Omega \to R$ given by $J(\alpha) = \int_0^1 \|\dot{\alpha}(t)\|^2 \, dt$.

Let us look at the "first variation formulae" of this calculus of variations problem defined by J. The space of variations of α, or the "tangent space" of Ω at α, is the linear space defined by

$$\{\eta \colon [0, 1] \to T(M) | \eta(t) \in T_{\alpha(t)}(M), \eta(0) = 0, \eta(1) = 0\}.$$

One can think of $T_\alpha(\Omega)$ as the space of vector fields along the curve α.

One can think of geodesics as being like "critical points" of $J \colon \Omega \to R$. It would be natural to define the derivative of J at $\alpha \in \Omega$, $DJ(\alpha) \colon T_\alpha(\Omega) \to R$ as follows.

In case M is R^n, then the tangent bundle $T(M)$ is $M \times R^n$ and one can proceed by letting $F \colon T(M) \to R$ be defined by $F(x, \dot{x}) = \|\dot{x}\|_x$ for $\dot{x} \in T_x$. Then for $\eta \in T_\alpha(\Omega)$ let

$$DJ(\alpha)(\eta) = \int_0^1 \frac{\partial F}{\partial x}\eta + \frac{\partial F}{\partial \dot{x}}\dot{\eta} \, dt,$$

where $\partial F/\partial x = (\partial F/\partial x)(\alpha(t), \dot{\alpha}(t))$ is a linear map on R^n for each t. Then if $DJ(\alpha) = 0$,

$$DJ(\alpha)(\eta) = \int_0^1 \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\eta = 0 \quad \text{for all } \eta \in T_\alpha(\Omega).$$

So Euler's equation for α is satisfied or:

Euler's equation: $d/dt \, \partial F/\partial \dot{x}(\alpha(t), \dot{\alpha}(t)) - \partial F/\partial x(\alpha(t), \dot{\alpha}(t)) = 0.$

If M is not R^n, then one has the argument and equation valid in each coordinate chart, or one could use the covariant derivative.

Recall that a curve $\alpha \colon [a, b] \to M$ is a *geodesic* if it locally minimizes length. Then the above kind of derivation shows that α in Ω is a geodesic if and only if α is a solution of Euler's equation in each coordinate chart. Thus the geodesics are indeed like critical points of J.

The earliest part of our review motivates the question as to whether there are the concepts of index and nullity for a geodesic α. In the "second derivative" of J at α, as before, one can find an answer. In a coordinate chart of M, it is natural to write for $D^2J(\alpha)$, the second derivative of J at α, the symmetric bilinear form on $T_\alpha(\Omega)$ defined by

$$D^2J(\alpha)(\eta, \xi) = H_\alpha(\eta, \xi) = \int_0^1 F_{xx}(\eta, \xi) + F_{x\dot{x}}(\eta, \dot{\xi}) + F_{\dot{x}x}(\dot{\eta}, \xi) + F_{\dot{x}\dot{x}}(\dot{\eta}, \dot{\xi}) \, dt.$$

Here, $\eta, \xi \in T_\alpha(\Omega)$ and $F_{xx} = F_{xx}(\alpha(t), \dot\alpha(t))$ is the second partial derivative of F as a bilinear form on $T_{\alpha(t)}(M) = R^n$, etc.

One defines the *index* and *nullity* of α simply as the index and nullity of H_α on $T_\alpha(\Omega)$. Furthermore, α is *nondegenerate* if H_α is.

Define an inner product on $T_\alpha(\Omega)$ via that on M: i.e.,

$$(\eta, \xi)_\alpha = \int_0^1 (\eta(t), \xi(t))_{\alpha(t)}\, dt, \qquad \eta, \xi \in T_\alpha(\Omega).$$

Using integration by parts, one obtains that for all η, ξ in $T_\alpha(\Omega)$,

$$H_\alpha(\eta, \xi) = (L\eta, \xi)$$

where

$$L\eta = -\frac{d}{dt}(F_{\dot x\dot x}\dot\eta + F_{x\dot x}\eta) + F_{xx}\eta - F_{x\dot x}\dot\eta$$

is the *Jacobi* (linear) *differential operator.*

One may express L invariantly in terms of the Riemann curvature tensor.

Say that P and Q are *conjugate* along α if $L\eta = 0$ for some nonzero $\eta \in T_\alpha(\Omega_{P,Q})$. The *multiplicity* of this conjugacy is the dimension of the linear space of such η.

The *Morse index theorem* asserts that the index of α is equal to the number of points $\alpha(t)$, $0 < t < 1$, such that $\alpha(t)$ is conjugate to $\alpha(0)$ along α, counting conjugate points with multiplicity. I like to think of this result as belonging to the spectral theory of differential operators.

To prepare us for his main result, Morse shows that for prescribed P in M, if Q is excluded from a set of measure zero in M, then all the geodesics in $\Omega_{P,Q}$ will be nondegenerate. Thus for such Q, J is like the Morse functions defined earlier. We may call such a pair (P, Q) a *nondegenerate pair.*

Now we may state the following basic theorem of Morse in the calculus of variations.

THEOREM. *Let (P, Q) be a nondegenerate pair on a complete Riemannian manifold M. Let B_i denote the dimension of the homology group $H_i(\Omega_{P,Q}(M))$ (over the rationals) of the loop space. Let M_i denote the number of geodesics joining P to Q (in Ω) of index i. Then the B_i, M_j satisfy the Morse inequalities*

$$M_0 \geqslant B_0, \qquad M_1 - M_0 \geqslant B_1 - B_0, \quad etc.$$

as before.

Morse proves the theorem by taking a sequence of finite dimensional manifolds which approximate Ω and applying the earlier abstract theory. These approximating manifolds are manifolds of piecewise geodesic curves.

As a particular case of this theorem, Morse in his first book takes M to be homeomorphic to the n-sphere S^n. By applying the same Morse inequalities to the standard Riemannian n-sphere in R^{n+1}, he is able to compute the B_i, Betti numbers of the loop space, which are given by (say for $n > 2$), $B_i = 1$ for $i = 0, n - 1, 2(n - 1), 3(n - 2), \ldots$ and zero otherwise. Therefore, he concludes that for an arbitrary Riemannian structure on S^n, the existence of geodesics of index $0, n - 1, 2(n - 1), \ldots$ joining P to Q.

Applications of this basic theorem to a more general class of manifolds were hampered by lack of knowledge of the homology of loop spaces. In fact, the breakthrough on this problem didn't come until 1951 with Serre's thesis. Using the Leray spectral sequence, Serre showed that $H_i(\Omega(M)) \neq 0$ for a sequence of i going to infinity for a compact manifold M with finite fundamental group. From the Morse theorem, this is enough to conclude that on any compact Riemannian manifold, any nondegenerate pair (P, Q) is joined by an infinite number of geodesics.

The history of the problem of *closed* geodesics on a manifold homeomorphic to a sphere S^m is one with fine achievements; it is also a subject over which many mathematicians have stumbled. For example, in the foreword of his 1934 book, when discussing Poincaré's work on existence of closed geodesics on a convex surface, Morse says the validity of Poincare's reasoning "has been questioned". Explicit objections are presented by Morse in his Chapter 9. These last two chapters of his book in fact are devoted to showing the existence of $m(m + 1)/2$ closed geodesics of a special kind on a Riemannian manifold homeomorphic to S^m. Yet this work of Morse is in error, as was pointed out by A. S. Svarc in 1960. Bott in 1954 gave a new proof which again was in error as Svarc noted.

Some successes in this line have been accepted. Lusternik and Schnirelmann (1929) showed the existence of three closed geodesics without self-intersection on a surface homeomorphic to S^2. G. D. Birkhoff in 1927 showed that a manifold homeomorphic to S^n had at least one closed geodesic.

In the last two decades, these questions have been pursued with very substantial success, especially by Soviet and German mathematicians. Klingenberg in a recently printed set of notes (Bonn, 1976) gives an account of the subject of closed geodesics complete with history. These notes include a proof of his newest theorem: On every simply connected manifold there exist infinitely many closed (prime) geodesics. (Let us hope ....)

At this point, we add that two expositions of Morse theory have probably been much more widely read than Morse's original treatise. These are the books of Seifert and Threlfall and of Milnor. Especially for those who have learned their mathematics in recent decades, Milnor's book is to be recommended.

The language and concepts in the calculus of variations since 1736 (Euler) have suggested that extremals could be thought of as critical points of the functional, length, area, energy, etc. Our review emphasizes this analogy. In fact, this analogy is actual. The geodesics *are* critical points of the energy. One can put a manifold structure on Ω in such a way that the energy J becomes a differentiable map on Ω and an abstract Morse theory on infinite dimensional manifolds can be developed which yields Morse's theorems for geodesics. This is what Palais and I did in 1962–1964. One obtains a unification of the first and second parts of the material of our review. For example, a geodesic is literally a critical point, the definition of Hessian and index of an abstract critical point apply directly to give these notions for geodesics as a special case. One needs no longer to approximate Ω by finite dimensional manifolds of broken geodesics. The abstract theory already applies to Ω.

Two contributions had made the way easier for us. First, Eells had put an infinite dimensional manifold structure (locally Hilbert or locally Banach) on certain function spaces in 1958. Secondly, Lang did the foundations of differential topology for Banach manifolds in 1962. Then Ralph Abraham, Palais, and I, in 1962, worked out systematically some theory of manifolds of function spaces and properties of manifolds of function spaces.

The idea for this way of doing Morse theory is to replace the compactness condition of Ω by a compactness condition on the map J, a condition which Palais and I called *Condition* C.

More precisely, let Ω be a complete Riemannian manifold, no longer compact or even finite dimensional, but defined as before with coordinate charts as open sets in Hilbert space. Let J be a (smooth) function on Ω which is bounded below, has nondegenerate critical points (of finite index for simplicity) and satisfies:

CONDITION C. If α_i in Ω is a sequence with $J(\alpha_i)$ bounded and such that $\|DJ(\alpha_i)\|$ tends to zero, then α_i has a convergent subsequence.

The result is that for such a function, the Morse inequalities are true, relating the type numbers defined by critical points of J and the Betti numbers of Ω. The proof is essentially that given above via handle attaching.

Now for the Morse theory of geodesics, one only has to show that the energy J on the loop space Ω satisfies the above properties, with an appropriate manifold structure on Ω. Palais and I used the Sobolev completion of Ω, with norm defined by L^2 and first derivatives in L^2. Actually it was Palais who wrote out the theory for this case in his article in *Topology* in 1963.

Some of the new results on closed geodesics mentioned earlier were proved using Condition C as above.

Recently, Tony Tromba seems to have found a drastic modification of Condition C, and developed a Morse theory for geodesics using infinite dimensional manifolds with a space of much smoother curves.

Now I would like to spend the last few words of this review on the global variational calculus for more than one independent variable. It was in fact just this problem that led me into infinite dimensional manifolds in the early sixties. It remains a problem; although very recently, especially through the work of Tromba and Karen Uhlenbeck, there seem to be signs of progress.

The most interesting case for more than one independent variable is minimal surfaces. In the theory of Plateau's problem, I had been intrigued by a result of Morse-Tompkins and Shiffman in 1939. Their theorem asserted that if a Jordan curve in R^3 spans two stable minimal surfaces, then it spans a third of unstable type. This was suggestive of a Morse theory for Plateau's problem.

In the sixties I tried without success to find such a theory, or to imbed the Morse-Tompkins-Shiffman result in a conceptual general setting. Tromba and Uhlenbeck may now have succeeded in initiating a development of calculus of variations in the large for more than one independent variable.

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Rings of quotients, by Bo Stenström, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 217, Springer-Verlag, New York, Heidelberg, Berlin, 1975, viii + 309 pp., $39.60.

Noncommutative ring theorists have long been tantalized by the method of localization used so easily and successfully by their commutative colleagues. It is unfortunate, yet typical, that results and techniques which are almost trivial for commutative rings turn out to be either false or impossible for noncommutative rings. Stenström's *Rings of quotients* records the attempts at developing a comprehensive, general technique of localization for noncommutative rings.

The study of quotient rings for noncommutative rings goes back to the early 1930s with the question in van der Waerden's first edition about whether noncommutative integral domains could be embedded in division rings. Ore, in 1931, found a criterion (the "Ore condition") for an integral domain to have a division ring of fractions: Given nonzero elements a and b, there exist nonzero c and d such that $ac = bd$. Independently, Wedderburn, in 1932, proved directly, by a similar procedure, that Euclidean domains have division rings of fractions.

The subject attracted little interest until the early fifties. There was, however, an important development due to Asano [**1**]. Asano's result was of less interest than his method, both of which will be described here. If R is a commutative ring and S a multiplicatively closed subset of non-zero-divisors,