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A CHARACTERIZATION OF HARMONIC IMMERSIONS OF SURFACES

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Let S be an oriented surface with Riemannian metric ds^2, and M^n a Riemannian manifold of dimension $n \geqslant 2$. We present here a characterization of harmonic immersions $f: S \longrightarrow M^n$ which sheds some light on their differential geometric properties. While C^∞ smoothness is assumed throughout, less is needed.

To work on the Riemann surface determined by ds^2 on S, use conformal parameters $z = x_1 + ix_2$ which correspond to ds^2-isothermal coordinates x_1, x_2 on S. Given any local coordinates on M^n, write $f = (f^\alpha)$ and $f_i^\alpha = \partial f^\alpha/\partial x_i$ where $i = 1, 2$ and $\alpha, \beta, \gamma = 1, 2, \ldots, n$. An immersion $f: S \longrightarrow M^n$ is harmonic if and only if for each α and for any ds^2-isothermal coordinates x_1, x_2 on S

$$\partial^2 f^\alpha/\partial x_i^2 + \Gamma_{\beta\gamma}^\alpha f_i^\beta f_i^\gamma = 0,$$

where $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols for the metric on M^n, and one sums on the indices β, γ and i.

To any real quadratic form $X = l_{ij}dx_i dx_j$ on S, associate on R the quadratic differential $\Omega(X, R)$ and the conformal metric $\Gamma(X, R)$ given by $4\Omega(X, R) = (l_{11} - l_{22} - 2il_{12})dz^2$ and $2\Gamma(X, R) = (l_{11} + l_{22})dzd\bar{z}$ respectively. Thus $X = 2 \operatorname{Re} \Omega + \Gamma$ on R. (See [10].) Call $\Omega(X, R)$ holomorphic if and only if the coefficient of dz^2 is complex analytic in z for every conformal parameter z on R. An immersion $f: S \longrightarrow M^n$ yields many quadratic forms of interest, among them the induced metric I, and the second fundamental forms II(N) determined by choices of a unit normal vector field N.

DEFINITION. An immersion $f: S \longrightarrow M^n$ is *R-minimal* if and only if $\Omega(\mathrm{I}, R)$ is holomorphic, and $\Gamma(\mathrm{II}(N), R) \equiv 0$ for any choice (local or global) of a unit normal vector field N.

An R-minimal immersion is *minimal* if and only if R is the Riemann surface R_I determined on S by I. It is known that a conformal immersion $f: S \longrightarrow M^n$ is harmonic if and only if it is minimal. Indeed, this is established in [2] independent of the dimensions of S and M^n. By analogy, we have the following

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THEOREM. *An immersion $f: S \longrightarrow M^n$ is harmonic if and only if it is R-minimal.*

This result is known for maps $f: S \longrightarrow M^2$. (See [4] for references.) It is also known that $\Omega(\mathrm{I}, R)$ must be holomorphic for any harmonic map $f: S \longrightarrow M^n$, so that the only harmonic maps of the 2-sphere must be minimal ([2] and [8]). We consider immersions here to provide (when $n \geqslant 3$) a well-defined $(n-2)$-dimensional normal space everywhere.

Note that $\Gamma(\mathrm{II}(N), R) \equiv 0$ for all N means that the trace of $\mathrm{II}(N)$ with respect to ds^2 vanishes for all N. When $ds^2 \propto \mathrm{I}$, this condition alone forces a minimal immersion, for it says that the mean curvature vector [11, p. 13] vanishes. Indeed, by our Theorem, the "mean curvature vector" formed with ds^2 in place of I vanishes for any harmonic immersion $f: S \longrightarrow M^n$. The converse can fail when $R \neq R_{\mathrm{I}}$. For example, if S is immersed in E^3 with Gauss curvature $K \equiv -1$, the usual asymptotic Tchebychev coordinates [9, p. 528] are II$'$-isothermal, where $\sqrt{H^2+1}\,\mathrm{II}' = H\mathrm{II} + \mathrm{I}$, with H mean curvature. Here $\Gamma(\mathrm{II}, R_{\mathrm{II}'}) \equiv 0$ but $\Omega(\mathrm{I}, R_{\mathrm{II}'})$ is *not* holomorphic. Similarly, $\Omega(\mathrm{I}, R)$ holomorphic does not imply $\Gamma(\mathrm{II}(N), R) \equiv 0$ for any N. This is obvious when $R = R_{\mathrm{I}}$. Less trivially, if S is immersed in E^3 with $K \equiv 1$, then $\Omega(\mathrm{I}, R_{\mathrm{II}}) \not\equiv 0$ is holomorphic, but $\Gamma(\mathrm{II}, R_{\mathrm{II}}) \equiv \mathrm{II}$ does *not* vanish [5].

The proof of the theorem is elementary, using the Gauss equations [5, p. 160]. Some results which follow from the theorem are stated below for the special case $n = 3$. Full details and proofs will appear elsewhere. Hereafter, $f: S \longrightarrow M^3$ is an immersion with fundamental forms I and II, mean curvature H, Gauss curvature K and intrinsic curvature $K(\mathrm{I})$. Denote by $\mathcal{K}$ the sectional curvature of M^3 for planes tangent to S, by $\Lambda = g\mathrm{I} + h\mathrm{II}$ any *positive definite* linear combination with real valued coefficients g and h, by R the Riemann surface determined on S by ds^2 and by $\mathcal{R}$ an arbitrary Riemann surface on S. The form II$'$ given by $\sqrt{H^2 - K}\,\mathrm{II}' = H\mathrm{II} - K\mathrm{I}$ is positive definite wherever $K < 0$ [10]. Lemmas 1 and 2 reflect the separate effects of the conditions $\Omega(\mathrm{I}, R)$ holomorphic and $\Gamma(\mathrm{II}, R) \equiv 0$. Theorem 2 includes a correction of the Corollary to Theorem 2 in [7].

LEMMA 1. *If $\Omega = \Omega(\mathrm{I}, \mathcal{R}) \not\equiv 0$ is holomorphic, then except at isolated points where $\Omega = 0$, there exists a cannonically determined function $F > 0$ on S which is $\mathcal{R}$-superharmonic where $K(\mathrm{I}) \geqslant 0$ and $\mathcal{R}$-subharmonic where $K(\mathrm{I}) \leqslant 0$* [1, p. 135].

LEMMA 2. *If $\Gamma(\mathrm{II}, \mathcal{R}) \equiv 0$ for any one $\mathcal{R}$ on S, then $K \leqslant 0$ (so that $K(\mathrm{I}) \leqslant \mathcal{K}$), and $H = 0$ wherever $K = 0$.*

THEOREM 1. *If $f: S \longrightarrow M^3$ is harmonic with $ds^2 = \Lambda$, then either $\Lambda \propto \mathrm{I}$, or else (except at isolated points where $\Lambda \propto \mathrm{I}$) $\Lambda \propto \mathrm{II}'$.*

THEOREM 2. *If* $f\colon S \longrightarrow M^3$ *is harmonic with* $ds^2 = \mathrm{II}'$, *H never zero and* $0 \not\equiv K(\mathrm{I}) \leqslant 0$, *then* H'/H *is not bounded.*

THEOREM 3. *If* $f\colon S \longrightarrow M^3$ *is harmonic with* $ds^2 = \mathrm{II}'$ *complete,* $|K/H|$ *bounded and* $K(\mathrm{II}') \leqslant 0$ *then* $K(\mathrm{II}') \equiv 0$.

THEOREM 4. *If* $f\colon S \longrightarrow M^3$ *is harmonic with R parabolic* [1, p. 209], I *nowhere proportional to* ds^2 *and* $K(\mathrm{I}) \geqslant 0$, *then* $K(\mathrm{I}) \equiv 0$.

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