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SECOND ORDER ELLIPTIC EQUATIONS WITH MIXED BOUNDARY CONDITIONS

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We consider the mixed boundary value problem (MBVP) $Au = f$ in Ω, $B^+u = g^+$ in Γ^+, $u = g^-$ in Γ^- where Ω is a bounded open subset of R^n whose boundary Γ is divided into disjoint open subsets Γ^+ and Γ^- by an $(n-2)$-dimensional manifold ω in Γ. We assume $A = \Sigma_{|\alpha|\leqslant 2} a_\alpha(x)D^\alpha$ is a *properly elliptic* operator on $\overline{\Omega}$ and $B^+ = \Sigma_{j=1}^n b_j^+(x)D_j + b_0(x)$ is a *normal* boundary operator satisfying the *complementing condition* with respect to A on $\overline{\Gamma^+}$. The coefficients of the operators and Γ^+, Γ^- and ω are all assumed arbitrarily smooth.

Throughout, s will denote a real number with $s \not\equiv \frac{1}{2} (\mathrm{mod}\ 1)$. For $G = R^n$, $R^n_\pm$, Ω or Γ, the Sobolev spaces $H^s(G)$ are as in Lions-Magenes [1]. Also $H^s(\Gamma^\pm)$ is the space of restrictions to $\Gamma^\pm$ of distributions in $H^s(\Gamma)$, with the infimum norm, and $H^s_A(\Omega) = \{u \in H^s(\Omega): Au \in L^2(\Omega)\}$ with the graph norm. Let γ_0: $H^s_A(\Omega) \longrightarrow H^{s-1/2}(\Gamma)$ be the trace map, $r^\pm$: $H^{s-1/2}(\Gamma) \longrightarrow H^{s-1/2}(\Gamma^\pm)$ the restriction maps, and $\gamma^- = r^-\gamma_0$. Then $B^+ = r^+B$ for some first-order normal boundary operator B on the whole of Γ.

Consider the maps $(A, \gamma^-, B^+)_s$ defined as

$$(A, \gamma^-, B^+)\colon H^s(\Omega) \longrightarrow H^{s-2}(\Omega) \times H^{s-1/2}(\Gamma^-) \times H^{s-3/2}(\Gamma^+) \quad \text{if } s > 3/2,$$

$$(A, \gamma^-, B^+)\colon H^s_A(\Omega) \longrightarrow L^2(\Omega) \times H^{s-1/2}(\Gamma^-) \times H^{s-3/2}(\Gamma^+) \quad \text{if } s < 3/2.$$

These maps are bounded for all s, by the condition of normality for $s < 3/2$ (see for example [1, §2.8.1]). The MBVP is called *well-posed* if there exists $s \not\equiv \frac{1}{2} (\mathrm{mod}\ 1)$ for which $(A, \gamma^-, B^+)_s$ is Fredholm. A bounded linear operator between Hilbert spaces is called *α-semi-Fredholm* (αsF) if it has finite dimensional kernel and closed range, *β-semi-Fredholm* (βsF) if it has closed range with finite codimension, and *Fredholm* if it is αsF and βsF.

THEOREM. *For each $x \in \omega$ there is an open subset I_x of the reals such that for $s \not\equiv \frac{1}{2} (\mathrm{mod}\ 1)$, $(A, \gamma^-, B^+)_s$ is Fredholm if and only if $s \in I = \bigcap_{x\in\omega} I_x$. Moreover, I is open and so the MBVP is well-posed if and only if I is nonempty. In fact, for each $x \in \omega$ there is a real number e_x determined algebraically*

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by the coefficients of A and B^+ *at x such that–with* $e = \{r \in \mathbf{R}: r \equiv e_x (\mathrm{mod}\ 1)$ *for some* $x \in \omega\}$:

(a) *if* $n = 2$, $I_x = \{r \in \mathbf{R}: r \not\equiv e_x\ (\mathrm{mod}\ 1)\}$ *and, hence,* $I = R - e$;

(b) *if* $n = 3$, $I_x = (e_x, e_x + 1)$ *or* $I_x = \varnothing$, *and* $(A, \gamma^-, B^+)_s$ *is* αsF (βsF) *if and only if* $s \notin e$ *and* $s > \sup_{x \in \omega} e_x$ ($s < \inf_{x \in \omega} \tilde{e}_x$) *where* $\tilde{e}_x$ *is* $e_x + 1$ *in the first case, and* e_x *in the second*;

(c) *if* $n > 3$, $I_x = (e_x, e_x + 1)$ *and* $(A, \gamma^-, B^+)_s$ *is* αsF (βsF) *if and only if* $s \notin e$ *and* $s > \sup_{x \in \omega} e_x$ ($s < \inf_{x \in \omega} e_x + 1$).

Peetre [2] showed that $(A, \gamma^-, B^+)_s$ is Fredholm for $s > 3/2$ and $s \notin e$, without the assumption of normality, but only for $n = 2$. Shamir [5] and Visik and Eskin [6] provided elements of the solution of the MBVP for $n > 2$, but the restriction $s > 3/2$, required for nonnormal problems, and in addition, the problems of localising when $s < 3/2$, prevented them from finding necessary and sufficient conditions for the problem to be well posed in our sense.

For normal B^+, we can construct a bounded sesquilinear form $J^s[u, v]$ on a closed subspace $V^s(\Omega) \times W^{2-s}(\Omega)$ of $H^s(\Omega) \times H^{2-s}(\Omega)$ such that $(A, \gamma^-, B^+)_s$ is αsF (βsF) if and only if J^s is αsF (βsF) in the sense that the operator T^s: $V^s(\Omega) \longrightarrow W^{2-s}(\Omega)^*$ defined by $\langle T^s u, v\rangle = J^s[u, v]$ is αsF (βsF). This result is an easy consequence of Pryde [3].

By Peetre's lemma [1, Lemma 2.5.1] J^s is αsF if and only if

$$(1)\quad \|u\|_{V^s(\Omega)} \leqslant c\left(\sup_{v \in W^{2-s}(\Omega)} \frac{|J^s[u, v]|}{\|v\|_{W^{2-s}(\Omega)}} + \|u\|_{H^{s-1}(\Omega)}\right), \qquad u \in V^s(\Omega),$$

and βsF if and only if

$$(2)\quad \|v\|_{W^{2-s}(\Omega)} \leqslant c\left(\sup_{u \in V^s(\Omega)} \frac{|J^s[u, v]|}{\|u\|_{V^s(\Omega)}} + \|v\|_{H^{1-s}(\Omega)}\right), \qquad v \in W^{2-s}(\Omega).$$

The advantage of looking at forms is that their estimates can be localised for all s. For this we use *spaces with homogeneous norms,* $Z^s(G)$ and $\mathring{Z}^s(G)$ for $G = \mathbf{R}^n$ or $\mathbf{R}^n_\pm$. In fact, let $[u; G]_s$ be the norm on $C_0^\infty(\overline{G})$ defined by

(i) if $s = 0$, $[u; G]_s = \|u\|_{L^2(G)}$;

(ii) if $0 < s < 1$, $[u; G]_s = (\int_G \int_G |u(x) - u(y)|^2/|x - y|^{n+2s}\, dx\, dy)^{1/2}$;

(iii) if $s \geqslant 1$, $[u; G]_s = (\Sigma_{|\alpha| = [s]}\ [D^\alpha u; G]^2_{s-[s]})^{1/2}$ where $[s]$ is the integral part of s.

For $s \geqslant 0$, $Z^s(G)$ ($\mathring{Z}^s(G)$) is the completion of $C_0^\infty(\overline{G})$ ($C_0^\infty(G)$) with respect to $[u; G]_s$. For $s < 0$, $Z^s(G)$ ($\mathring{Z}^s(G)$) is the strong dual of $\mathring{Z}^{-s}(G)(Z^{-s}(G))$.

These spaces have been considered before, but their theory not fully developed. See, for example, Shamir [4]. They are the natural spaces for boundary value problems in $\mathbf{R}^n_+$ with homogeneous operators and constant coefficients.

For each $x \in \omega$, we freeze the coefficients of A and B^+ at x, and drop all lower order terms, obtaining homogeneous operators with constant coefficients A_x and B^+_x. Define $(\gamma^-, B^+)_{s,x}$ to be

$$(\gamma^-, B_x^+)\colon Z^s_{\ker A_x}(\mathbf{R}^n_+) \longrightarrow Z^{s-1/2}(\mathbf{R}^{n-1}_-) \times Z^{s-3/2}(\mathbf{R}^{n-1}_+)$$

where $X_{\ker A}$ denotes the kernel of an operator A in the space X.

Using the results of Pryde [3] repeatedly, and the usual localisation techniques, it follows that J^s is αsF (βsF) if and only if $(\gamma^-, B^+)_{s,x}$ is left invertible (onto) for each $x \in \omega$.

Following Peetre [2], we then construct certain Wiener-Hopf operators $r^+M^s_x\colon L^2(\mathbf{R}^{n-1}_+) \longrightarrow L^2(\mathbf{R}^{n-1}_+)$ whose symbols are determined by the coefficients of A_x and B_x. It follows that $(\gamma^-, B^+)_{s,x}$ is left invertible (onto) if and only if $r^+M^s_x$ is left invertible (onto).

Finally, using the results of Shamir [5], we find real numbers e_x, $\widetilde{e}_x$ and open sets I_x as above, such that $r^+M^s_x$ is an isomorphism if and only if $s \in I_x$. Moreover, if $n \geqslant 3$, $r^+M^s_x$ is left invertible (onto) if and only if $s > e_x$ $(s < \widetilde{e}_x)$ and $s \not\equiv e_x (\mathrm{mod}\ 1)$. If $n > 3$, $\widetilde{e}_x = e_x + 1$ and if $n = 3$, $\widetilde{e}_x = e_x + 1$ or e_x.

Detailed proofs and corresponding results for higher order operators will appear in another paper. The work was part of a Ph.D. thesis at Macquarie University. I am indebted to my supervisor Alan McIntosh.

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