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TWO-POINT PADÉ TABLES AND T-FRACTIONS[1]

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Continued fractions of the form

$$1 + d_0 z + \frac{z}{1 + d_1 z} + \frac{z}{1 + d_2 z} + \cdots, \qquad d_n \in \mathbf{C}, \tag{1}$$

called *T*-fractions, were introduced by one of the authors in 1948 [7]. He showed that *corresponding* to a given formal power series (fps)

$$L = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \tag{2}$$

at $z = 0$, there exists a unique *T*-fraction (1) with the property that the Taylor expansion about $z = 0$ of the nth approximant of (1) agrees with (2) up through the term $c_n z^n$. He further showed that every *T*-fraction *corresponds* to some fps (2) in the above sense. However, it is known that if all $d_n \neq 0$, then the approximants of the *T*-fraction expansion of L are not in the Padé table of L.

Perron [5, p. 179] was the first to observe that every *T*-fraction (1), with all $d_n \neq 0$ for $n \geqslant 0$, *corresponds* to some formal Laurent series (fLs)

$$L^* = c_{-1}^* z + c_0^* + \frac{c_1^*}{z} + \frac{c_2^*}{z^2} + \cdots, \qquad c_{-1}^* \neq 0, \tag{3}$$

at $z = \infty$ in the sense that the Laurent expansion of the nth approximant of (1) agrees with (3) up through the term c_{n-1}^*/z^{n-1} (see also [4], [8]). Attempts to relate the continued fraction expansion (1) at ∞ to that at 0 have been unsuccessful until now.

The concept of the Padé table [3] has recently been generalized to give rational approximants for formal Newton series called Newton-Padé approximants (see, for example, [2]) and for approximation alternately at 0 and ∞ called two-point Padé approximants (see, for example, [1], [6]). In this paper we show that the approximants of the *T*-fractions with all $d_n \neq 0$ for $n \geqslant 0$ are the $(n + 1, n)$ entries in the two-point Padé table of the series (2) and (3) to which the *T*-fraction corresponds. In addition we are able to give an explicit formula for the d_n in terms of the c_n and c_m^* (see Equation (6)) and also an explicit formula for the

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relation between the c_n and c_m^* (see Equation (5)). Our principal result is given by the following:

THEOREM. (A) *Suppose that a T-fraction* (1) *with* $d_n \neq 0$ *for all* $n \geqslant 0$ *corresponds* (*in the sense described above*) *to formal series* L *at* $z = 0$ *and* L^* *at* $z = \infty$ *of the forms* (2) *and* (3), *respectively. Then the following conditions must hold*:

$$\Delta_n \neq 0, \quad n \geqslant 0, \tag{4}$$

$$\Omega_{n+1} = \Delta_n + (-1)^n c_{-1}^* \Delta_{n-1}, \quad n \geqslant 0, \tag{5}$$

$$d_0 = c_{-1}^*, \quad d_1 = \frac{1}{\Delta_1}, \quad d_n = \frac{\Delta_{n-1}^2}{\Delta_n \Delta_{n-2}}, \quad n \geqslant 1, \tag{6}$$

where

$$\Delta_{-1} = \Delta_0 = 1, \quad \Delta_1 = c_0^* - 1,$$

$$\Delta_2 = \begin{vmatrix} c_1^* & (c_0^* - 1) \\ (c_0^* - 1) & (c_{-1}^* - c_1) \end{vmatrix},$$

$$\Delta_n = \begin{vmatrix} c_{n-1}^* & \cdots & c_1^* & (c_0^* - 1) \\ \vdots & \iddots & \iddots & (c_{-1}^* - c_1) \\ \vdots & \iddots & \iddots & -c_2 \\ c_1^* & \iddots & \iddots & \vdots \\ (c_0^* - 1) & (c_{-1}^* - c_1) & -c_2 \cdots & -c_{n-1} \end{vmatrix}, \quad n \geqslant 1$$

$$\Omega_1 = c_1,$$

$$\Omega_2 = \begin{vmatrix} c_1 & (c_0^* - 1) \\ c_2 & (c_{-1}^* - c_1) \end{vmatrix},$$

$$\Omega_{n+1} = \begin{vmatrix} c_1 & c_{n-1}^* & \cdots & c_1^* & (c_0^* - 1) \\ \vdots & \vdots & \iddots & \iddots & (c_{-1}^* - c_1) \\ \vdots & \vdots & \iddots & \iddots & -c_2 \\ c_{n-1} & c_1^* & \iddots & \iddots & \vdots \\ c_n & (c_0^* - 1) & \iddots & \iddots & \vdots \\ c_{n+1} & (c_{-1}^* - c_1) & -c_2 & \cdots & -c_n \end{vmatrix}, \quad n \geqslant 0$$

(B) *Suppose for given* $\{c_n\}$ *and* $\{c_n^*\}$ *that* (4) *and* (5) *are satisfied. Then the T-fraction* (1) *with* d_n *defined by* (6) *satisfies* $d_n \neq 0$ *for* $n \geqslant 0$ *and corresponds both to* L *and to* L^*. *The nth numerator* $A_n(z)$ *and denominator* $B_n(z)$ *of* (1) *have degrees exactly equal to* $n + 1$ *and* n, *respectively; hence the nth* *approximant $A_n(z)/B_n(z)$ of the T-fraction is the $(n+1, n)$ entry in the two-point Padé table of L and L^*.*

Finally, we note that a full two-point Padé table of the type described above always exists. If $R_{m,n} = P_{m,n}/Q_{m,n}$ is the (m, n) entry of the table, with $P_{m,n}$ and $Q_{m,n}$ relatively prime polynomials of degrees at most m and n, respectively, then

$$\text{(9a)} \qquad \text{if } m > n, \quad \text{degree } P_{m,n} \leqslant \begin{cases} \left[\frac{m-n}{2}\right], & \text{if } \left[\frac{m-n}{2}\right] \geqslant n+1, \\ n+1, & \text{if } \left[\frac{m-n}{2}\right] \leqslant n+1, \end{cases}$$

$$\text{(9b)} \quad \text{if } m \leqslant n+1, \quad \text{degree } Q_{m,n} \leqslant \begin{cases} \left[\frac{n-m}{2}\right], & \text{if } \left[\frac{n-m}{2}\right] \geqslant m-1, \\ m-1, & \text{if } \left[\frac{n-m}{2}\right] \leqslant m-1. \end{cases}$$

Here $[r]$ denotes the largest integer less than or equal to r. A proof of these statements and of the theorem will be included in a subsequent paper.

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