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FOURIER ANALYSIS ON COMPACT SYMMETRIC SPACE

BY THOMAS O. SHERMAN

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1. Let $L \supset K$ be Lie groups with complex Lie algebras $\mathfrak{l}_c$ and $\mathfrak{k}_c$. Assume $\mathfrak{k}_c$ has a linear complement $\mathfrak{b}$ in $\mathfrak{l}_c$ which is a subalgebra. For any σ in $\mathrm{LieHom}_{\mathbf{C}}(\mathfrak{b}, \mathbf{C})$ there is a unique germ of a C^ω function e^σ at $s_0 := K$ in $S := L/K$ such that $e^\sigma(s_0) = 1$ and $xe^\sigma = \sigma(x)e^\sigma$ (x in $\mathfrak{b}$). Now suppose S is connected, K is compact, and e^σ extends to an element of $C^\omega(S)$. Then (Harish-Chandra) $\varphi_\sigma(s) := \int_K e^\sigma(ks)\,dk$ is a spherical function in the sense that

$$\int_K \varphi_\sigma(gks)\,dk = \varphi_\sigma(gK)\varphi_\sigma(s).$$

For a Riemannian symmetric space of noncompact type Helgason [1], [2] extended Harish-Chandra's spherical transform theory to a Fourier theory in which functions of the form e^σ mimic the role of characters in classical Fourier theory on $\mathbf{R}^n$. Here we report that difficulties inherent in copying these ideas over to compact symmetric space have been overcome, at least for the rank one spaces.

2. Let $S := U/K$ be symmetric with U compact semisimple. Let G_c be a complexification of U and G a noncompact real form of G_c such that $K_0 := G \cap U$ is open in K, and maximal compact in G. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition and set $\mathfrak{b} := \mathbf{C}(\mathfrak{a} + \mathfrak{n})$. Then $\mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{b}$ as in §1. Λ will denote the set of those λ in $\mathrm{LieHom}_{\mathbf{C}}(\mathfrak{b}, \mathbf{C})$ such that e^λ is in $C^\omega(S)$. $\Lambda | i\,\mathfrak{a}$ is the set of highest restricted weights of K-spherical representations of U. For λ in Λ let V_λ denote the corresponding irreducible U-submodule of $L^2(S)$. Then e^λ is the highest weight vector in V_λ. Define τ in $\mathrm{LieHom}_{\mathbf{C}}(\mathfrak{b}, \mathbf{C})$ by $\tau(x) := \mathrm{tr}(\mathrm{ad}\, x|\mathfrak{b})$ (x in $\mathfrak{b}$). Then τ is in Λ.

LEMMA 1. *There is a unique maximal connected, open, K-invariant neighborhood S_0 of s_0 in S on which $e^\tau \neq 0$. Then $e^\lambda \neq 0$ on S_0 for all λ in Λ.*

On S_0 define $e_*^\lambda := (e^{\lambda+\tau})^{-1}$. e_*^λ is the inverse transform kernel to e^λ. The aforementioned "inherent difficulty" of the subject is the singularity of e_*^λ off of S_0. Let $B := K/M$ where M is the centralizer of $\mathfrak{a}$ in K.

LEMMA 2. *For all uK in S_0, s in S, and λ in Λ*

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$$\int_B e^\lambda(k^{-1}s)e_*^\lambda(k^{-1}uK)\,dkM = \varphi_\lambda(u^{-1}s).$$

PROOF. While this result may be proved directly in the full generality of §1 it follows in the present case by analytic continuation via G_c of the similar result of Helgason on G/K_0 (e.g. middle of p. 116, [1]). [5] contains related analytic continuation arguments and helpful machinery linking U/K and G/K_0.

Let $L_c^2(S_0) := \{f \in L^2(S_0) \mid \operatorname{supp}(f) \text{ is compact in } S_0\}$ and let $d_\lambda := \dim(V_\lambda)$. Lemma 2 combines with well-known harmonic analysis on S (see e.g. [3, Chapter 10]) to give

THEOREM 1. *For f_1 in $L_c^2(S_0)$ define F_*f_1 on $B \times \Lambda$ by*

$$F_*f_1(kM, \lambda) := \int_{S_0} f_1(s)e_*^\lambda(k^{-1}s)\,ds.$$

Then $\Sigma_\Lambda d_\lambda \int_B F_*f_1(kM, \lambda)e(k^{-1}s)\,dkM \longrightarrow f_1(s)$ (*in* $L^2(S_0)$).

THEOREM 2. *For f_2 in $L^2(S)$ define Ff_2 on $B \times \Lambda$ by*

$$Ff_2(kM, \lambda) := \int_S f_2(s) \operatorname{conj.}(e^\lambda(k^{-1}s))\,ds.$$

Then $\Sigma_\Lambda d_\lambda \int_B Ff_2(kM, \lambda)\operatorname{conj.}(e_*^\lambda(k^{-1}s))\,dkM \longrightarrow f_2(s)$ (*in* $L^2(S)$).

THEOREM 3. *For f_1 in $L_c^2(S_0)$, f_2 in $L^2(S)$*

$$\int_{S_0} f_1(s) \operatorname{conj.}(f_2(s))\,ds = \Sigma_\Lambda d_\lambda \int_B F_*f_1(b, \lambda) \operatorname{conj.}(Ff_2(b, \lambda))\,db.$$

3. To extend these results from S_0 to S we must give global definitions of e_* and F_*. This is done for S of rank one as follows. Let λ_1 be the generator (over $\mathbf{Z}^+$) of Λ. Let $\rho := \operatorname{Re}(e^{\lambda_1})$ except $\rho := 1$ if $S = P_d(\mathbf{R})$. Then

$$e_*^\lambda(s) := (\operatorname{sgn}(\rho(s)))^{1+\dim S}(e^{\lambda+\tau}(s))^{-1} \qquad (s \text{ in } S).$$

Where $\mu(s)$ is the distance on S from s_0 to s let

$$S(\alpha, \beta) := \left\{s \in S \mid |\rho(s)| \geqslant \alpha,\ \mu(s) \leqslant -\beta + \sup_{x\in S} \mu(x)\right\}.$$

Then for f in $C^\infty(S)$,

$$F_*f(kM, \lambda) := \lim_{\beta\to 0^+} \lim_{\alpha\to 0^+} \int_{S(\alpha,\beta)} f(s)e_*^\lambda(k^{-1}s)\,ds$$

defines a distribution on S, at least for the rank one symmetric spaces, and if we replace S_0 by S and $L_c^2(S_0)$ by $C^\infty(S)$ in Theorems 1 and 3 they continue to hold. Theorem 2 may also be made global by carefully defining the order of integration over B. This has been carried out in detail for the sphere in [4].

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DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MASSACHUSETTS 02115