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THE TRANSFER AND COMPACT LIE GROUPS

BY MARK FESHBACH

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1. **Introduction.** A map $\rho\colon X \longrightarrow Y$ between two spaces induces a homomorphism $\rho^*\colon h(Y) \longrightarrow h(X)$ between the cohomology groups of the spaces, where h is an arbitrary cohomology theory. In certain situations a transfer homomorphism $\tau^*\colon h(X) \longrightarrow h(Y)$ has been defined by Becker and Gottlieb, Dold and others. The compositions $\tau^* \circ \rho^*\colon h(Y) \longrightarrow h(Y)$ and $\rho^* \circ \tau^*\colon h(X) \longrightarrow h(X)$ are of considerable interest as they relate the cohomologies of X and Y. The first type of composition is relatively easy to compute. The second is, in general, quite difficult.

Let G be a compact Lie group with H and K arbitrary closed subgroups with associated l-universal classifying spaces BG, BH, BK. Let $\rho(H, G)\colon BH \longrightarrow BG$ be the natural projection. Then transfers $T(H, G)\colon h(BH) \longrightarrow h(BG)$, $T(K, G)\colon h(BK) \longrightarrow h(BG)$ are defined by Dold's definition where $T(H, G) = T_{\mathrm{id}}^{BH}$ in Dold's notation [D]. The main theorem is a double coset type theorem which generalizes the classical double coset theorem for finite groups [C–E, p. 257]. It is proved for arbitrary compact Lie groups.

2. **Main result.** Let $K|G|H$ be the double coset space obtained as the orbit space of the left action of K on G/H. This space breaks up into a finite disjoint union of orbit-type manifold components $\{M_i\}$. Let $g_i \in G$ be a representative of M_i. Let $\chi^\#(M_i) = \chi(\overline{M}_i) - \chi(\overline{M}_i - M_i)$ be the internal Euler characteristic of M_i. Then if $H^g = gHg^{-1}$ we have

THEOREM 1 (DOUBLE COSET).

$$\rho^*(K, G) \circ T(H, G) = \sum \chi^\#(M_i) T(H^{g_i} \cap K, K) \circ \rho^*(H^{g_i} \cap K, H^{g_i}) \circ Cg_i$$

where the sum is over the orbit-type manifold components of $K|G|H$. $Cg\colon h(BH) \longrightarrow h(BH^g)$ is the cohomology isomorphism induced by the obvious map from BH^g to BH.

This theorem holds where G is a compact Lie group and H and K are arbitrary closed subgroups.

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[1]Based on a thesis written at Stanford University under the supervision of Professor G. Brumfiel.

Theorem 1 simplifies in special cases, e.g. when it is known that $\chi^{\#}(M_i) = 0$. In particular, the following easily proved result is often useful.

THEOREM 2. *Assume $N_G(H)/H$ is not discrete where $N_G(H)$ is the normalizer of H in G. Then $T(H, G) = 0$.*

3. Examples. In the case where G is finite the double coset space $K|G|H$ is discrete. Since $\chi^{\#}(\text{pt.}) = 1$ we recover the classical double coset theorem for finite groups.

Among many other corollaries to Theorem 1 we have

THEOREM 3. *Let G be a compact Lie group with H any closed subgroup of G. Let K be an arbitrary torus in G. Then*

$$\rho^*(K, G) \circ T(H, G) = \sum \chi^{\#}(M_i)\rho^*(K, H^{g_i}) \circ Cg_i$$

where the sum is over the manifold components of the fixed point set.

THEOREM 4. *Let $G(n) = U(n)$, $H(n) = \Sigma_n \wr U(1)$ the wreath product of the permutation group on n letters Σ_n and $U(1)$. Let $K(n) = U(n-1)$. Then the diagram*

$$\begin{array}{ccc} BH(n-1) & \xrightarrow{\pi(n-1)} & BH(n) \\ \downarrow & & \downarrow \\ BU(n-1) & \xrightarrow{\rho(n)} & BU(n) \end{array}$$

satisfies the following stability condition

$$\rho^*(n) \circ T(H(n), U(n)) = T(H(n-1), U(n-1)) \circ \pi^*(n-1).$$

A similar formula for the orthogonal groups appears in printed version of a talk by J. C. Becker [B].

THEOREM 5. *Let $G(n+m) = U(n+m)$, $H(n+m) = \Sigma_{n+m} \wr U(1)$, $K(n, m) = U(n) \times U(m)$. Then the diagram*

$$\begin{array}{ccc} BH(n) \times BH(m) & \xrightarrow{\pi} & BH(n+m) \\ \downarrow & & \downarrow \\ BU(n) \times BU(m) & \xrightarrow{\rho} & BU(n+m) \end{array}$$

satisfies the following relation

$$\rho^* \circ T(H(n+m), U(n+m)) = T(H(n) \times H(m), U(n) \times U(m)) \circ \pi^*.$$

Both Theorems 4 and 5 are proven by using Theorem 2 to simplify the double coset formula to a single term.

4. Proof. The proof of Theorem 1 is geometric and involves constructing a K-equivariant deformation of the identity map of G/H by appealing to the covering homotopy theorem of Palais [**Br**, p. 97]. Elementary properties of the transfer are then used. The proof with minor modifications applies to more general situations.

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201