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ALGEBRAIC SURFACES AND IRRATIONAL CONNECTED SUMS OF FOUR MANIFOLDS

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Suppose W is an irreducible nonsingular projective algebraic 3-fold and V a nonsingular hypersurface section of W. Denote by V_m a nonsingular element of $|mV|$. Let V_1, V_m, V_{m+1} be generic elements of $|V|$, $|mV|$, $|(m+1)V|$ respectively such that they have normal crossing in W. Let $S_{1m} = V_1 \cap V_m$ and $C = V_1 \cap V_m \cap V_{m+1}$. Then S_{1m} is a nonsingular curve of genus g_m and C is a collection of $N = m(m+1)V_1^3$ points on S_{1m}. By [MM2] we find that

$$(*) \qquad V_{m+1} \text{ is diffeomorphic to } \overline{V_m - T(S_{1m})} \cup_\eta \overline{V_1' - T(S_{1m}')}$$

where $T(S_{1m})$ is a tubular neighborhood of S_{1m} in V_m, V_1' is V_1 blown up along C, S_{1m}' is the strict image of S_{1m} in V_1', $T(S_{1m}')$ is a tubular neighborhood of S_{1m}' in V_1' and $\eta: \partial T(S_{1m}) \longrightarrow \partial T(S_m')$ is a bundle diffeomorphism.

Now V_1' is well known to be diffeomorphic to $V_1 \# N(-CP^2)$ (the connected sum of V_1 and N copies of CP^2 with opposite orientation from the usual). Thus in order to be able to inductively reduce questions about the structure of V_m to ones about V_1 we must simplify the "irrational sum" (*) above.

The general question we can ask is then the following:

Suppose M_1 and M_2 are compact smooth 4-manifolds and K is a connected q-complex embedded in M_i. Let T_i be a regular neighborhood of K in M_i and let $\eta: \partial T_1 \longrightarrow \partial T_2$ be a diffeomorphism:

Set $V = \overline{M_1 - T_1} \cup \overline{M_2 - T_2}$. How can the topology of V be described more simply in terms of those of M_1 and M_2?

In those notes we announce some results on this question for the case $q = 1, 2$ and indicate some applications to the topology of algebraic surfaces. Let $\approx$ be read as "is diffeomorphic to" and set $P = CP^2$ and $Q = -CP^2$. Then

THEOREM A. *Let M_i, T_i, η be as above and suppose K is a wedge of k 1-spheres homotopic to zero in M_i. Suppose also η is orientation-reversing (relative to the induced orientation on T_i from M_i in case M_i is oriented) and identity-like (see definition below).*

Then either

(a) $V \approx M_1 \# M_2 \# k(S^2 \times S^2)$, *or*

(b) $V \approx M_1 \# M_2 \# k(P \# Q)$.

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(In case $M_1 \# M_2$ is of odd type then by [W] both (a) and (b) hold.)

We note that ∂T_i in the above theorem is a connected sum of $S^1 \times S^2$ with a 'collapsing' map c_i: $\partial T_i \longrightarrow K$ inherited from $\tilde{c}_i$: $T_i \rightarrow K$. We call a collection $\{e_j\}$ of 1-spheres in ∂T_i a 'nice geometrical basis' of ∂T_i iff there exists a collection Σ_j of 2 spheres in ∂T_i, generating $H_2(\partial T_i; \mathbf{Z})$ such that $c(\Sigma_j)$ is a point and $e_l \cap \Sigma_j$ is empty if $l \neq j$ and is a transversal intersection in one point if $l = j$. Then a diffeomorphism η: $\partial T_1 \longrightarrow \partial T_2$ is called identity-like if it is isotopic to a map transforming 'nice geometrical basis' to 'nice geometrical basis'.

Now suppose $q = 2$ and K is an embedded 2-manifold. Then

THEOREM B. *Let M_1, M_2 above be simply connected and suppose K above is an embedded 2-manifold with the T_i tubular neighborhoods of K in M_i. Let η: $\partial T_1 \longrightarrow \partial T_1$ be a bundle-diffeomorphism.*

Let C be a fiber of $\partial T_1 \longrightarrow K$ considered as a loop in V. Let k = minimal number of generators of $H_1(K, \mathbf{Z})$.

Then

(1) *If C is homotopic to zero in V then either*

$$V \# (S^2 \times S^2) \approx M_1 \# M_2 \# k(S^2 \times S^2)$$

or

$$V \# (P \# Q) \approx M_1 \# M_2 \# k(P \# Q)$$

with both alternatives holding if $M_1 \# M_2$ is of odd type.

(2) *If M_2 is obtained by blowing a manifold N up by a σ-process at a point of a submanifold S whose strict image in M_2 is K then $V \# P \approx M_1 \# N \# k(P \# Q)$.*

Theorem B part (2) is of particular interest in applications to algebraic surfaces. In particular in the example we gave at the beginning of this note we can conclude that

$$(**) \qquad V_{m+1} \approx V_m \# V_1 \# (N-1)Q \# 2g(S_{1m})(P \# Q)$$

(where $g(S_{1m})$ is the genus of the nonsingular curve S_m). In particular if we take $W = CP^3$ and V_1 a hyperplane of W then (**) above implies the main theorem of [MM1], which states that if V is any nonsingular hypersurface of CP^3 then $V \# P$ is diffeomorphic to a connected sum of P's and Q's.

We can apply Theorem B to get the following type of result.

Let us call a simply connected 4-manifold M completely decomposable iff for some integers k, $M \approx kP \# lQ$.

Then we have

THEOREM C. *Suppose X is a nonsingular algebraic surface such that $X \# P$ is completely decomposable. Let C be an irreducible nonsingular hypersurface* *section of X. Let $W \xrightarrow{\pi} X$ be the projective bundle over X obtained by compactifying the line bundle $E = [kC]$ $(k \geq 0)$.*

Then

(1) *If V is an irreducible, nonsingular hypersurface section of W then $V \# P$ is completely decomposable.*

(2) *If V is a nonsingular irreducible subvariety of W such that $\pi|V \longrightarrow X$ is an m-fold branched cover of X then $V \# P$ is completely decomposable.*

(3) *If V is a nonsingular k-fold cyclic branched cover of X with branch locus R linearly equivalent to kC for some $k > 0$ then $V \# P$ is completely decomposable.*

In particular the nonsingular 'double-planes' (i.e., 2-fold branched covers of CP^2) are all completely decomposable after taking their connected sum with CP^2.

REFERENCES

[MM1] R. Mandelbaum and B. Moishezon, *On the topological structure of non-singular algebraic surfaces in CP^3*, Topology 15 (1976), 23–40.

[MM2] ———, *On the topology of algebraic surfaces*, Topology (to appear).

[W] C. T. C. Wall, *Diffeomorphisms of 4-manifolds*, J. London Math. Soc. 39 (1964), 131–140. MR 29 #626.

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