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THE KUNZE-STEIN PHENOMENON

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ABSTRACT. We show that, if G is a connected semisimple Lie group with finite center, then $L^p(G) * L^2(G) \subseteq L^2(G)$ if $1 \leq p < 2$.

The theorem announced in the abstract was proved by R. A. Kunze and E. M. Stein [4] for the case when G is SL(2, **R**). Various authors have extended to various other groups G, but the methods used have been quite technical and hard to generalise. The difficulty is the expression of the analytic continuation of the principal series as uniformly bounded representations on a Hilbert space. We avoid this difficulty by treating isometric representations on mixed $L^{\mathbf{p}}$-spaces. Here is an outline of our method. We suppose that G has real rank r.

Let $\overline{N}MAN$ be a Bruhat decomposition of G, and $\alpha_1, \ldots, \alpha_r$ the associated simple positive roots. We denote by $\overline{N}_j$ the subgroup of $\overline{N}$ whose Lie algebra is the sum of the root spaces $\mathfrak{N}_{-\alpha}$, where

$$\alpha = m_j\alpha_j + m_{j-1}\alpha_{j-1} + \cdots + m_1\alpha_1,$$

with $m_j > 0$. By ρ_j we denote the element of the dual of the Lie algebra of A, defined by the rule

$$\rho_j(a) = -\tfrac{1}{2}\,\mathrm{tr}[\mathrm{ad}(a)|_{\overline{\mathfrak{N}}_j}],$$

and by ρ the sum of the ρ_j. The group $\overline{N}$ has a decomposition $\overline{N} = \overline{N}_r \cdots \overline{N}_1$. Almost all elements of G have a Bruhat decomposition: we write

$$g = \overline{N}(g)M(g)A(g)N(g).$$

The unitary class-one principal series can be realised on $L^2(\overline{N})$ by the formula

$$[\pi_z(g)\xi](\overline{n}) = \exp[-(\rho + z_1\rho_1 + \cdots + z_r\rho_r)\log A(g^{-1}n)]\,\xi(\overline{N}(g^{-1}n)),$$

where z is a purely imaginary r-tuple. Allowing z to be complex in the above formula, we obtain one "analytic continuation of the principal series" (see Stein [5]). Let $L^{\mathbf{p}}(\overline{N})$ be the space of functions on $\overline{N}$ such that the norm $\|\ \|_{\mathbf{p}}$:

$$\|\xi\|_{\mathbf{p}} = \left[\int_{N_r} dn_r \cdots \left[\int_{N_1} dn_1 |\xi(n_r \cdots n_1)|^{p_1}\right]^{p_2/p_1} \cdots\right]^{1/p_r},$$

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is finite. Then we have the following theorem:

THEOREM 1. *Suppose that* $p_j \operatorname{Re}(z_j) = 2 - p_j$. *Then the representation* π_z *acts isometrically on* $L^{\mathbf{p}}(\bar{N})$.

According to C. S. Herz [3], to prove the convolution theorem ($L^p(G) * L^2(G) \subseteq L^2(G)$) it suffices to show that, for ξ and η in $L^2(\bar{N})$, the function $g \longmapsto \langle \pi_0(g)\xi, \eta\rangle$ belongs to $L^{p'}(G)$, or, equivalently, that

$$|\langle \pi_0(u)\xi, \eta\rangle| \leqslant C\|u\|_p, \qquad u \in L^1 \cap L^p(G).$$

We fix ξ and η in $L^2(\bar{N})$ of norm one. For z in the tube T of r-tuples with $|\operatorname{Re}(z_j)| \leqslant 1$, we define ξ_z by the formulae:

$$\xi_z(\bar{n}) = \xi(\bar{n}) \prod_1^r \xi_j(\bar{n}) \quad \text{where} \quad \xi_1(\bar{n}) = |\xi(\bar{n})|^{z_1},$$

and if $j > 1$,

$$\xi_j(\bar{n}) = \left[\int_{\bar{N}_j} d\bar{n}_j \cdots \int_{\bar{N}_1} d\bar{n}_1 \, |\xi(\bar{n}\bar{n}_j \cdots \bar{n}_1)|^2\right]^{(z_j - z_{j-1})/2};$$

η_z is defined similarly, but $\bar{\eta}$ replaces ξ and $-z_j$ replaces z_j. With these definitions, ξ_z has norm one in $L^{\mathbf{p}}(\bar{N})$ and η_z has norm one in the conjugate space. Thus, for u in $L^1(G)$ and z in T we have the estimate

$$|\langle \pi_z(u)\xi_z, \eta_z\rangle| \leqslant \|\pi_z(u)\xi_z\|_{\mathbf{p}} \|\eta_z\|_{\mathbf{p}} \leqslant \|u\|_1 \|\xi_z\|_{\mathbf{p}} \|\eta_z\|_{\mathbf{p}} = \|u\|_1. \tag{1}$$

From the Plancherel formula for the class-one principal series (Harish-Chandra [1], [2] and G. Warner [6, Chapter 9]) we obtain the estimate

$$\begin{aligned} &\left[\int_{\mathbf{R}^r} dy \, |C(y)^{-1} \langle \pi_{iy}(u)\xi_{iy}, \eta_{iy}\rangle|^2\right]^{1/2} \\ &\qquad \leqslant \left[\int_{\mathbf{R}^r} dy |C(y)^{-1} \|\pi_{iy}(u)\|_{HS}|^2\right]^{1/2} \leqslant \|u\|_2 \end{aligned} \tag{2}$$

for u in $L^1 \cap L^2(G)$. From the known properties of the C-function, it follows that there are some nonzero linear functionals H_j on $\mathbf{R}^r$ and a constant C such that

$$\prod_1^k |\varphi(iH_j(y))| \leqslant C|C(y)|^{-1},$$

where $\varphi(z) = z/2 - z$. We now write $P_z(u)$ instead of $\langle \pi_z(u)\xi_z, \eta_z\rangle$, and our estimates (1) and (2) yield

$$|P_z(u)| \leqslant \|u\|_1, \qquad u \in L^1(G), \tag{3}$$

$$\left[\int_{\mathbf{R}^r} dy | \prod_1^k \varphi(iH_j(y)) P_{iy}(u)|^2\right]^{1/2} \leqslant C\|u\|_2. \tag{4}$$

We note that $P_z(u)$ depends analytically on z, by the construction of π_z, ξ_z, and η_z. The main theorem is proved by applying the following interpolation theorem, whose proof is a mild inductive variation of Kunze and Stein's Theorem 4 and Lemma 26.

THEOREM 2. *Suppose that $z \mapsto P_z$ is a weak-star topology analytic map of T into $L^\infty(G)$, and that the estimates* (3) *and* (4) *hold. Then, if* $1 \leqslant p < 2$,

$$|P_0(u)| \leqslant C(C, p, H_1, \ldots, H_k)\|u\|_p, \qquad u \in L^1 \cap L^p(G).$$

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