

DISCONTINUOUS HOMOMORPHISMS FROM $C(X)$

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Let X be an infinite compact Hausdorff space, let $C(X)$ denote the algebra of continuous complex-valued functions on X , and let $|\cdot|_X$ be the uniform norm on X . We announce two completely independent proofs of the following theorem.

THEOREM. *Assuming the continuum hypothesis (CH), there exists a discontinuous monomorphism from $(C(X), |\cdot|_X)$ into certain Banach algebras, and there is an incomplete algebra norm on $C(X)$.*

The conjecture that every algebra norm $\|\cdot\|$ on $C(X)$ is equivalent to the uniform norm arises naturally from a theorem of Kaplansky in 1949 that necessarily $\|f\| \geq |f|_X$ ($f \in C(X)$): see [9, 10.1]. The seminal paper on the automatic continuity of homomorphisms from $C(X)$ is the 1960 paper of Bade and Curtis [2] in which, for example, it is proved that there is a discontinuous homomorphism from $C(X)$ if and only if there is a radical homomorphism, a nonzero homomorphism from a maximal ideal of $C(X)$ into a commutative radical Banach algebra. In 1967, it was proved by Johnson that every homomorphism from certain noncommutative C^* -algebras is continuous: see [9, 12.4]. Sinclair proved recently that the existence of a discontinuous homomorphism is equivalent to the existence of an algebra norm on $C(X)/I$ for some nonmaximal prime ideal I of $C(X)$ [9, 11.7], and this was proved independently in [4]. It follows from the work of each of the present authors that such a norm exists provided $|C(X)/I| = \aleph_1$. Assuming CH, such an ideal exists for each X , and every nonmaximal prime ideal has this property if X is separable, but, if X is not separable, there may exist a prime ideal J such that $C(X)/J$ is not normable [4].

The work of the first author is contained in [3]. Write C for the continuous real-valued functions on $\beta\mathbb{N}$, let $\mathfrak{p} \in \beta\mathbb{N} \setminus \mathbb{N}$, let $M_{\mathfrak{p}} = \{f \in C : f(\mathfrak{p}) = 0\}$, let $J_{\mathfrak{p}} = \{f = 0 \text{ near } \mathfrak{p}\}$, and let $A = M_{\mathfrak{p}}/J_{\mathfrak{p}}$. Then the quotient field of A is a real-closed totally ordered η_1 -field of cardinality 2^{\aleph_0} , and hence is a nonstandard model of the reals. For $\sigma \geq 1$, let $\Omega_{\sigma} = \{\operatorname{Re} z > 1, |z| < \sigma\}$, let $A_{\sigma} = C^*(\overline{\Omega}_{\sigma}) \cap \hat{O}(\Omega_{\sigma})$, and let $A_{\infty} = \operatorname{ind} \lim A_{\sigma}$: A_{∞} is an algebra of germs of analytic functions on 'half-neighborhoods' of infinity. The major part of [3] is the construction of a morphism $A \rightarrow A_{\infty}$. At one point, a recent result of

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Johnson [8] is used: two real-closed η_1 -fields of cardinality \aleph_1 are isomorphic as algebras over \mathbf{R} . For $\sigma > 0$, let $L_\sigma = \{f: \int_0^\infty |f(t)| e^{-\sigma t} dt < \infty\}$, and let $\Lambda = \bigcup L_\sigma$. The inverse Laplace transform is a monomorphism from the image of A to Λ , and Λ can be embedded in certain radical Banach algebras, including $C_*[0, 1]$, the space $C[0, 1]$ with convolution multiplication. This algebra does not have a bounded approximate identity. Sinclair [10] shows that there is a monomorphism from Λ , and hence from A , into any commutative radical Banach algebra with bounded approximate identity.

The work of the second author does not use Johnson's theorem. In [5], a 'formal power series algebra' B_{ω_1} is constructed using very old results of Hahn and Hausdorff, and it is proved that, assuming CH, the existence of a real algebra norm on B_{ω_1} implies the existence of a discontinuous homomorphism from $C(X)$ for each infinite X . Let C'_{ω_1} be the unique maximal ideal of C_{ω_1} , the complexification of B_{ω_1} . It is proved in [6] that a commutative radical Banach algebra B contains a copy of C'_{ω_1} if and only if there exists an embedding ψ of a suitable semigroup T'_{ω_1} in B such that $\psi(d + d') = \psi(d)\psi(d')$ and $\overline{\psi(d)B} = \overline{\psi(d')B}$ ($d, d' \in T'_{\omega_1}$). The proof uses some techniques introduced by Allan in [1] and general results of valuation theory. Such an embedding is constructed in [7] for each nonunital commutative Banach algebra with bounded approximate identity. So for every commutative radical Banach algebra B with bounded approximate identity and every infinite X there exists a discontinuous homomorphism from $C(X)$ into $B \oplus \mathbf{C}e$.

The existence of an algebra norm on C_{ω_1} does not depend on CH, and the construction of the first author is valid, independent of CH, if A is replaced by B_{ω_1} . However, if $2^{\aleph_0} > \aleph_1$, there are real-closed fields of cardinality 2^{\aleph_0} whose algebra of infinitesimals cannot be embedded in B_{ω_1} . Also, there exist models of set theory (ZFC \sim CH) such that, for each $\mathfrak{p} \in \beta\mathbf{N} \setminus \mathbf{N}$, $M_{\mathfrak{p}}/J_{\mathfrak{p}}$ cannot be embedded in B_{ω_1} . We do not know whether or not there exists an incomplete algebra norm on $C(X)$ if CH is not assumed.

Our methods lead easily to the existence of a discontinuous homomorphism from a Banach algebra with infinitely many characters.

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