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SINGULAR INVARIANT EIGENDISTRIBUTIONS AS CHARACTERS

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1. Let G be a connected, acceptable linear real semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. We assume that $\operatorname{rank}(K) = \operatorname{rank}(G)$, and we let T be a Cartan subgroup of G contained in K. We denote by $\mathfrak{G}_{\mathbf{C}}$ and $\mathfrak{t}_{\mathbf{C}}$ the complexifications of the Lie algebras of G and T respectively. The character group of T may be identified with a lattice L_T in the dual of $\sqrt{-1}\,\mathfrak{t}$, and the Weyl group $W_{\mathbf{C}}$ of the pair $(\mathfrak{G}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ acts on L_T. We say that $\lambda \in L_T$ is *regular* if $w\lambda \neq \lambda$ for all $w \neq 1$ in $W_{\mathbf{C}}$; otherwise λ is said to be *singular*. We denote the set of singular λ by L_T^s.

To each $\lambda \in L_T$, Harish-Chandra has associated a tempered invariant eigendistribution $\Theta(\lambda)$ on G ([1], [2]), and, if λ is regular, $\Theta(\lambda)$ is (up to a sign) a discrete series character of G. Our interest is centered on the distributions $\Theta(\lambda)$, λ singular, which we call *singular invariant eigendistributions associated to T.* More generally, we consider a class of singular invariant eigendistributions associated to any conjugacy class of Cartan subgroups of G.

The singular invariant eigendistributions mentioned above appear in the explicit formula for the Fourier transform of certain orbital integrals on G (see [3], [8]). The goal of this note is the character theoretic identification of these singular distributions. We first use a result of Zuckerman [12] which states that the tempered invariant eigendistributions on G which are "limits of discrete series" are actually characters on G. Then, we embed these characters in unitary principal series representations of G by appealing to a theorem of Hirai [5] which, for a restricted class of real simple Lie groups, characterizes those tempered invariant eigendistributions which are uniquely determined by their restriction to a distinguished Cartan subgroup in their support. In a recent note [4], the first author has removed the restrictions on G and has proved Hirai's theorem for any connected, acceptable, reductive Lie group with compact center.

In the case when G has split rank equal to one, the results announced in this note were worked out in part several years ago with K. Okamoto, and, more recently, in complete detail with N. Wallach. We note that our work overlaps with the recent work of Schmid [9] and Knapp and Zuckerman [7], but both our motivation and our techniques of proof are different.

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2. We first consider $\Theta(\lambda)$, $\lambda \in L_T^s$. Let F^+ be a connected component of

$$F' = \{\lambda \in \sqrt{-1}\, t^* \mid w\lambda \neq \lambda \quad \text{for all} \quad w \neq 1,\ w \in W_{\mathbf{C}}\}.$$

If $\lambda \in L_T^s \cap \overline{F^+}$, we define $\Theta(\lambda, F^+)$, the "limit of discrete series" (from F^+), as in [1] and [2]. From [2], we know that

$$\Theta(\lambda) = [W(\lambda)]^{-1} \sum_{w \in W(\lambda)} \Theta(\lambda, wF^+), \tag{2.1}$$

where $W(\lambda) = \{w \in W_{\mathbf{C}} \mid w\lambda = \lambda\}$. It follows from Zuckerman's results that, up to a sign, $\Theta(\lambda, wF^+)$ is a unitary tempered character on G.

If $\lambda \in L_T^s$ and λ is fixed by a nontrivial element of W_K, the Weyl group of K, then $\Theta(\lambda)$ is identically zero on G. Now fix $\lambda \in L_T^s$ such that $w\lambda \neq \lambda$ for all $w \neq 1$ in W_K. Associated to the set of roots of $(\mathfrak{G}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ which are orthogonal to λ is a Cartan subgroup J of G whose split part we denote by $J_{\mathfrak{p}}$. Pick $P \in P(J_{\mathfrak{p}})$, the collection of parabolic subgroups of G whose split part is $J_{\mathfrak{p}}$, and let $P = MJ_{\mathfrak{p}}N$ be a Langlands decomposition of P. Then $J_M = M \cap J$ is a compact Cartan subgroup of M, and, suitably interpreted, λ is a regular character on J_M^0 (^{0} denotes the connected component). Let $T(\lambda)$ be the discrete series character on M^0 corresponding to λ, pick a character χ on J_M compatible with $T(\lambda)$ and a character μ on $J_{\mathfrak{p}}$. If $M^+ = J_M M^0$, then $T(\lambda, \chi) = \mathrm{Ind}_{M^+ \uparrow M}\, \chi \otimes T(\lambda)$ is a discrete series character of M, and the induced character $\Theta(\lambda, \chi, \mu) = \mathrm{Ind}_{P \uparrow G} T(\lambda, \chi, \mu)$ is a unitary principal series character of G which is irreducible if μ is regular (see [11]) and may be reducible if μ is not regular.

THEOREM 2.2. *There is a character* χ_0 *of* J_M *such that the induced character* $\Theta(\lambda, 0, \chi_0)$ *is reducible, and*

$$\Theta(\lambda, 0, \chi_0) = \sum_{w \in W(\lambda)} (-1)^{q_M} \det(w)\, \Theta(\lambda, wF^+),$$

where $q_M = \tfrac{1}{2}\dim(M/K \cap M)$ *and* F^+ *is chosen so that* $\epsilon(F^+) = 1$ *(see* [2]).

It follows from [6] that $\Theta(\lambda, 0, \chi_0)$ has at most $[W(\lambda)]$ irreducible components.

COROLLARY 2.3. $(-1)^{q_M}\det(w)\Theta(\lambda, wF^+)$ *is an irreducible character and* $\Theta(\lambda, 0, \chi_0)$ *has exactly* $[W(\lambda)]$ *irreducible components.*

3. Now let $P = MAN$ be a cuspidal parabolic subgroup of G and T_M a compact Cartan subgroup of M. If λ is a singular character on T_M, we denote by $T(\lambda)$ the associated singular invariant eigendistribution on M^0. Let χ be a character on T_M which is compatible with $T(\lambda)$ and μ a character on A. As in (2.1), we may write $T(\lambda)$ on M^0 as a sum of limits of discrete series characters, and $T(\lambda) = 0$ if $w\lambda = \lambda$ for some $w \neq 1$ in $W(M^0, T_M^0)$. We assume that $T(\lambda) \neq 0$.

In the usual way, we write

$$\Theta(\lambda, \mu, \chi) = \mathrm{Ind}_{P\uparrow G}\, T(\lambda, \mu, \chi)$$

and

$$\Theta(\lambda, wF^+, \mu, \chi) = \mathrm{Ind}_{P\uparrow G}\, \mu \otimes T(\lambda, wF^+, \chi).$$

Again, using Zuckerman's results, we have the character theoretic decomposition

$$\Theta(\lambda, \mu, \chi) = [W(\lambda)]^{-1} \sum_{w\in W(\lambda)} \Theta(\lambda, wF^+, \mu, \chi). \tag{3.1}$$

Now, applying the procedure of §2, we arrive at a parabolic subgroup $P_1 = M_1A_1N_1$ and a compact Cartan subgroup J_{M_1} of M_1 such that λ, suitably interpretated, is a regular character on $J^0_{M_1}$. Let $T_1(\lambda)$ be the discrete series character on M_1^0 corresponding to λ, take a character χ_1 on J_{M_1} compatible with $T_1(\lambda)$ and extend μ trivially to A_1 ($A \subseteq A_1$). Then, define $R(\lambda, \mu, \chi_1) = \mathrm{Ind}_{P_1\uparrow G}\, T_1(\lambda, \mu, \chi_1)$ as in §2.

THEOREM 3.2. *Let $W_0 = W(\lambda) \cap W(M, T_M)$. Then there exists a character χ_1 on J_{M_1} (depending on χ) such that*

$$[W_0]\, R(\lambda, \mu, \chi_1) = \sum_{w\in W(\lambda)} (-1)^{q_{M_1}} \det(w)\, \Theta(\lambda, wF^+, \mu, \chi).$$

REFERENCES

1. Harish-Chandra, *Discrete series for semisimple Lie groups.* I, Acta Math. 113 (1965), 241–318. MR 36 #2744.

2. ———, *Two theorems on semi-simple Lie groups,* Ann. of Math. (2) 83 (1966), 74–128. MR 33 #2766.

3. R. A. Herb, *Fourier inversion on semisimple Lie groups of real rank two,* Thesis, Univ. of Washington, 1974.

4. ———, *A uniqueness theorem for tempered invariant eigendistributions*, Pacific J. Math. 67 (1967).

5. T. Hirai, *Invariant eigendistributions of Laplace operators on real simple Lie groups.* III (to appear).

6. A. W. Knapp, *Commutativity of interwining operators.* II, Bull. Amer. Math. Soc. 82 (1976), 271–273.

7. A. W. Knapp and G. Zuckerman, *Classification of irreducible tempered representations of semisimple Lie groups*, Proc. Nat. Acad. Sci. U.S.A. 73 (1976), 2178–2180.

8. P. J. Sally and G. Warner, *The Fourier transform on semisimple Lie groups of real rank one,* Acta Math. 131 (1973), 1–26.

9. W. Schmid, *On the characters of the discrete series,* Invent. Math. 30 (1975), 47–144.

10. G. Warner, *Harmonic analysis on semisimple Lie groups,* Vols. I, II, Springer-Verlag, Berlin and New York, 1972.

11. J. A. Wolf, *Unitary representations on partially holomorphic cohomology spaces,* Mem. Amer. Math. Soc. No. 138 (974).

12. G. Zuckerman, *Tensor products of infinite-dimensional and finite-dimensional representations of semisimple Lie groups* (to appear).

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