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Representations of commutative semitopological semigroups, by Charles F. Dunkl and Donald E. Ramirez, Lecture Notes in Mathematics, no. 435, Springer-Verlag, Berlin, Heidelberg, New York, 1975, 181 + vi pp., $8.60.

It is probably unnecessary to say that a *semigroup* is a set with an associative multiplication; yet it may be useful to state in the beginning that a *topological semigroup* is a semigroup equipped with a topology making multiplication jointly continuous, and that a *semitopological semigroup* differs from it insofar as the multiplication may be only separately continuous in each variable. A reader who is somewhat familiar with topological groups but less acquainted with semigroups may wonder about the necessity of this distinction; it seems to play a small role in group theory. The reason is that, for the most commonly treated types of groups such as those on locally compact or Polish (completely metrizable 2nd countable spaces), separate continuity of multiplication implies the axioms of a topological group (Ellis, Effros); thus the distinction is largely unnecessary for groups. For semigroups, however, the only class other than groups showing a similar behavior is that of *semilattices*, i.e. commutative semigroups in which all elements are idempotent; J. D. Lawson recently showed that every compact semitopological semilattice must be topological. (In this general context, he proved that every subgroup of a compact semitopological semigroup is in fact a topological group; this fails for a subsemilattice.) But in general, separate continuity on semigroups does not imply joint continuity. Moreover, semitopological semigroups which are not topological arise quite naturally in great variety, notably in analysis. Indeed in many instances semigroups occur here as semigroups of bounded operators on a Banach space; only if one considers the operator norm topology will such semigroups be automatically topological; in the more commonly considered operator topologies such as the strong or weak (or, in the case of Hilbert spaces, ultraweak) topology they will be semitopological, but rarely topological. It is therefore only natural that the functional analysts, notably when they prepare to talk on representation theory, should immediately turn to semitopological semigroups, as do Dunkl and Ramirez as soon as they give us the title of their book.

For numerous reasons the study of topological and semitopological semigroups has a different flavor from the investigation of topological groups. It is much more recent and much less developed than the latter, despite the existence of considerable journal literature on the subject spread over the last quarter century. Even when we compare compact groups and semigroups, the most striking difference is the absence of invariant integration on semigroups; some compact semigroups have no invariant measure (on either side); and in fact no compact semigroup which is not a group has an invariant measure whose support is the whole semigroup. Consequently, L^2-representations of semigroups are rare and not as organic as in the case of groups. Other phenomena compound this difficulty and make finite dimensional linear representations sparse: If we look at the unit interval under the multiplication $(x,y) \mapsto \min\{x,y\}$ (a standard compact semilattice called the *min-interval*), then we recognize the impossibility of nonconstant finite dimensional representations by the fact that every chain of projections on a finite dimensional space is finite, hence discrete; alternatively, the interval $[\frac{1}{2}, 1]$ under the multiplication $(x,y) \mapsto \max\{xy, \frac{1}{2}\}$ (a quotient of the ordinary multiplicative unit interval, or *standard interval*, obtained by squashing $[0, \frac{1}{2}]$ to a point) is a topological semigroup without any nonconstant finite dimensional linear representation, since every element different from the identity is nilpotent. This semigroup, therefore, is called the *nil-thread*. (We note in passing that locally, near the identity, the nil-thread is indistinguishable from the standard interval, which is one of the tamest compact semigroups in existence.) After such elementary examples it should be plausible why a coherent general structure and representation theory is unavailable even for compact topological semigroups–in any case by comparison with what is known about compact groups. Yet we do have a certain body of general theory, however narrow, for topological semigroups (and the interested nonexpert may find some of it in an article on topological semigroups I wrote recently for a general audience [**5**]); but in relation to this knowledge, the field of semitopological semigroups (even in the compact case) appears frightfully forbidding, not only through the absence of almost any general theory, but also through the immense complications occurring even in the abelian case in naturally arising semigroups such as the structure space of a convolution measure algebra, or the ergodic theory of a single operator on a Banach space. When West, and then Brown and Moran, uncovered some of the striking features of compact *semitopological* semigroups generated by one element (such as the presence of several (indeed uncountably many)) idempotents, about which Dunkl and Ramirez also inform us in their book, this was certainly a surprise for people knowing the relatively simple structure of singly generated compact *topological* semigroups. In view of the awe-inspiring difficulties with semitopological semigroups it is certainly a reasonable tactic on the part of Dunkl and Ramirez to restrict their attention to commutative semigroups while they are breaking ground on the harmonic analysis of semitopological semigroups.

Semigroups occur wherever one looks (as Hille remarked in the introduction of his pioneering book on *Functional analysis and semigroups* in 1948); much has been written about them, but comparatively little of a comprehensive, monograph or book type character. In a recent review in this journal [**2**] of the book by Balbes and Dwinger on distributive lattices, Mary Katherine Bennett takes note of a widespread scepticism towards lattice theory due to the ubiquity of lattices and the presence of trivial literature recognizable as such by the nonexpert. I fear that some of the same sentiment is in the air about semigroup theory, and topological semigroup theory, too. Such global criticism, both here and there, is certainly not based on profound competence nor intimate knowledge of all the mathematics in those areas; for there are deep, difficult and interesting results and problems in either of these fields, and both lines link up with many different branches of mathematics. It is true, research on both lattices and semigroups is primarily theory-oriented while the trend of the time begins to favor problem-oriented endeavors again; but neither excludes the other, and I find speculation on their relative merit idle. However, external pressures on the profession in a period of mathematical recession in the seventies have caused the number of dissertations written in topological or analytical semigroup theory to become retrograde by comparison with their boom in the fifties and sixties. Under such circumstances it has to be noted gratefully that Dunkl and Ramirez, armed with a bunch of pretty good new ideas, join an uphill fight, and win a battle with their monograph. They demonstrate that commutative harmonic analysis is alive on semigroups and full of possibilities and challenging problems–if such proof were really necessary after Taylor's herculean work on convolution measure algebras; and indeed Taylor's influence on the monograph under review is evident.

Before we have a closer look at Dunkl's and Ramirez' book we pause for a moment to glance at the existing monograph literature (a more detailed tabulation and evaluation is presented in [**5**]). The block of monographs concerned with topological semigroups and their structure comprises the introductory treatise to topological semigroups by Paalman de Miranda, 1964 [**10**]; the tour de force treatment of the theory up to the middle sixties by Hofmann and Mostert, 1966 [**8**]; a discussion of duality and finite dimensional representation of compact topological semigroups of Hofmann, 1970 [**6**]; a systematic evaluation of character duality of compact semilattices and its applications by Hofmann, Mislove and Stralka, 1974 [**7**]; and there will be a new book by Carruth, Hildebrant and Koch, 1977 (?) [**4**] which promises to become the new handbook on topological semigroups. Except for the 1970 duality treatise, there is little if any emphasis on harmonic analysis in all this literature. By contrast, we have a second block of monographs now being enlarged by the present one, which is concerned with analysis and with semitopological semigroups: A survey and treatise on weakly almost periodic functions and the associated compactifications by Berglund and Hofmann, 1967 [**1**], a hardcover book with similar intent by Burkel, 1970 [**3**]; and one may count Taylor's monograph on measure algebras, 1972 [**12**] in this collection because semitopological compact semigroups play an important role in it. As complementary reading in this line one should recommend Williamson's then comprehensive survey on harmonic analysis on semigroups, 1967 [**13**] and the article by Rothman and Schuh on Laplace transforms on vanishing algebras, 1974 [**11**]. The reader desirous to see this literature placed in a more historical frame may wish to consult [**5**]. Dunkl and Ramirez make every effort to make their monograph self-contained by presenting the required background material in introductory sections and appendices; only the reader who wishes to study the proofs of the prerequisites will have to consult some of the above source material and other references on harmonic analysis or operator theory such as Rudin's and Sakai's books.

What are Dunkl and Ramirez up to, what is novel in their approach? In a nutshell, harmonic analysis is concerned with certain classes of representations; one associates with a representation its "coefficient functions" and all functions arising in this fashion are called representative functions. Let us recall for one moment how this works for compact groups and the class of finite dimensional linear representations: Suppose S is a compact group and $s \mapsto T_s\colon S \mapsto B(E)$ a continuous linear representation on a finite dimensional complex vector space E. Let $R(S)$ be the set of all functions $s \mapsto \langle T_s u, v\rangle$ with $u \in E$ and $v \in E^*$, the dual of E. This is the same thing as taking all functions $s \mapsto \langle T_s, \omega\rangle$, with $\omega \in B(E)^*$ $(= B(E)_*)$, where the lower star denotes the predual of a dual space. Since we can form direct sums and tensor products of finite dimensional representations without leaving the class, $R(S)$ is an algebra, and since we can pass to the adjoint representation, it is even conjugate closed; according to the famous theorem of Peter and Weyl it is sup norm dense in $C(S)$. The theory of finite dimensional representations and the concomitant theory of $R(S)$ has been extended to compact semigroups where things become more complicated due to the fact that $R(S)$ may fail to be conjugate closed (since averaging, hence unitarisation, is no longer available); what is worse: We earlier saw some simple examples for which there are no nonconstant finite dimensional representations, whence $R(S)$ contains only the constant functions. This theory is accessible in [**6**], but much remains to be done.

For a commutative semitopological semigroup S, Dunkl and Ramirez begin by first choosing a new class of representations: They consider all ultraweakly continuous semigroup morphisms $s \mapsto T_s\colon S \to A_1$ into the unit ball of some commutative W^*-algebra A; since A is the dual of some Banach space A_*, the unit ball is weak-star compact, and the weak-star topology $\sigma(A, A_*)$ is the ultraweak topology. The associated set $R(S)$ of representative functions contains precisely the functions $s \mapsto \langle T_s, \omega\rangle$, where $\omega \in A_*$ is an element of the predual of the range algebra of A of some representations $T \in \mathbb{S}$. In fact, the authors proceed equivalently, but a bit more concretely and consider probability measure spaces (μ, Ω) and representations of S into the unit ball of $L^\infty(\mu, \Omega)$; the representative functions are then given by $s \mapsto \int T_s f\, d\mu$, $f \in L^1(\mu, \Omega)$. By observing the direct sum and the tensor product of commutative W^* algebras or, alternatively (as the authors do), simple measure space constructions, one notes that $R(S)$ is an involutive subalgebra of $C(S)$ (the algebra of bounded continuous functions on S). In fact, $R(S)$ is shown to be a subalgebra of the C^*-subalgebra $WAP(S)$ of $C(S)$ of all weakly almost periodic functions. The question of possible norms on $R(S)$ is a more delicate matter. The authors show in fact that the only case that $R(S)$ is closed in $C(S)$ in the sup-norm topology is that of finite dimension of $R(S)$. In particular (except for the case of finite dimensions), we have $R(S) \neq WAP(S)$. But a suitable norm can be found and for this norm, $R(S)$ is an involutive Banach algebra.

For each commutative semitopological semigroup S there is a commutative W^*-algebra, which I will call $W^*(S)$ and a representation $\rho\colon S \to W^*(S)_1$

such that for any ultraweakly continuous semigroup morphism $T: S \to A_1$ into the unit ball of a W^*-algebra there is a unique W^*-morphism (ultraweakly continuous *-morphism) $T': W^*(S) \to A$ such that $T'\rho = T$. (In functorial parlance this says that the functor $A \mapsto A_1$ (with the ultraweak topology on the unit ball A_1) from the category of W^*-algebras into the category of commutative semitopological semigroups has a left adjoint $W^*(\cdot)$.) The kernel congruence of ρ identifies precisely those pairs $s, t \in S$ which cannot be separated by any representation T, and these are indeed precisely those pairs which are not separated by any representative function from $R(S)$. (Recall the example of the nil-thread in which every element other than the identity was nilpotent; clearly $R(S)$ and $W^*(S)$ are one dimensional in this case, and ρ is constant!) If we denote with $W_*(S)$ the predual of $W^*(S)$, and with $\rho_*: W_*(S) \to R(S)$ the function defined by $\rho_*(\omega)(s) = \langle \rho(s), \omega \rangle$, then ρ_* is a quotient map of Banach spaces whose kernel is the annihilator in $W_*(S)$ of $\rho(S)$ and hence of the closed linear span of $\rho(S)$ in $W^*(S)$, which therefore is precisely the dual $R(S)^*$ of $R(S)$. Thus $R(S)^*$ is a function algebra as a closed subalgebra of $W^*(S)$, which generates $W^*(S)$ as a W^*-algebra (since $\rho(S)$ already does), and the corestriction $\rho: S \to R(S)^*$ is given by $\langle f, \rho(s) \rangle = f(s)$. In particular, if $\rho(S)$, hence $R(S)^*$, is conjugate closed, then $R(S)^* = W^*(S)$ and $R(S) = W_*(S)$.

When we now search for classes of commutative semitopological semigroups for which $R(S)^* = W^*(S)$ we discover a class which plays an important role in the theory; Dunkl and Ramirez call it class $\mathfrak{U}$; in order to describe which objects they collect in this class we observe that in any commutative semigroup the set $E(S)$ of idempotents is a semilattice (certainly nonempty if S is compact) and $H(S)$, the set of all s for which there is an s' with $ss' \in E(S)$, is a subsemigroup which is a union of groups; in the abelian situation these are precisely the so called inverse semigroups, and some people follow Mostert and myself in calling them Clifford semigroups, since A. H. Clifford was the first to provide a systematic treatment for them (1941). In a topological compact semigroup, $E(S)$ is closed, hence a compact semilattice, and $H(S)$ is closed, forming a compact topological Clifford semigroup, and in any such every maximal subgroup is compact. None of this persists in the semitopological case. Dunkl and Ramirez say that a semitopological abelian semigroup S belongs to class $\mathfrak{U}$ iff $S = H(S)^-$; even in the compact case this is still a remarkably large class contrary to what the topological case may lead one to believe. For example, the WAP-compactification of any abelian topological group is of this form. But, even more interestingly, if (μ, Ω) is a measure space with a *continuous* probability measure, then the unit ball $S = L^\infty(\mu, \Omega)_1$ is of class $\mathfrak{U}$. This yields an example of an S on which $x \mapsto x^2$ is not continuous and $E(S)$ is not closed: the constant function with value $\frac{1}{2}$ can be approximated by a net of characteristic functions. It is perhaps useful to note that this does not say that $E(S)$ fails to be a topological semilattice in the induced topology: in fact, it is. Every unit ball of a W^*-algebra embeds into another one which is a $\mathfrak{U}$-semigroup, and this applies particularly to

$W^*(S)$. Thus an S embeds into a unit ball of a W^*-algebra which is a $\mathfrak{A}$-semigroup iff $R(S)$ separates the points. $\mathfrak{A}$-semigroups are pretty closely looked at in the monograph. They are among those for which $R(S)^* = W^*(S)$, $R(S) = W_*(S)$. If $S \in \mathfrak{A}$, then $\|f\|_{R(S)} = \|f\|_{R(H(S))}$; a commutative inverse semigroup $S = H(S)$ has an involution $x \mapsto x'$ (with the unique inverse x' of x such that $xx' \in E(S)$). This allows the introduction of positive definite functions on S: indeed $f \in C(S)$ will be positive definite iff for every finite sequence $x_1, \ldots, x_n \in S$ the matrix $(f(x_i x_j'))_{i,j=1,\ldots,n}$ is the coefficient matrix of a positive semidefinite sesquilinear form on $\mathbf{C}^n$. But even if S is only of type $\mathfrak{A}$, this definition is possible if we restrict the choice of the x_i to $H(S)$. Dunkl and Ramirez then show as a core result a theorem relating to a group classic by Gelfand and Raikov: Let S be a commutative $\mathfrak{A}$-semigroup with identity. Then $f \in C(S)$ is positive definite iff there is a L^∞-representation $T\colon S \to L^\infty(\mu, \Omega)_1$ and some $g \in L^1(\mu, \Omega)$ with $g \geqslant 0$ such that $f(s) = \int T_s g \, d\mu$ for all $s \in S$. The key, as usual, is the concept that positive definite functions go together with Hilbert space representations.

In all of this, one naturally asks the question on the scope of the theory of representations into the unit ball of W^*-algebras and the corresponding theory of $R(S)$. Firstly, one wishes to see some examples; secondly one would like to know whether one has general criteria to ascertain that $R(S)$ separates the points. If S is a locally compact abelian group, then $R(S)$ identifies with $M(\hat{G})\hat{}$. If S is a subsemigroup of a locally compact abelian group G and S has uniformly positive Haar measure (i.e. $\int fh \, dm_g = 0$ for all $h \in L^1(G) \cap M(S)$ and $f \in C(S)$ iff $f = 0$), then $R(S) = R(G)|S$. For the theory of $L^1(S)$ for a subsemigroup of an l.c.a. group we may refer the reader to the article by Rothman and Schuh for further information [17]. These are the examples we have in the vicinity of groups. Otherwise, the authors compute $R(S)$ for the min-interval S and find that $f \in C(S)$ is in $R(S)$ iff it has bounded variation; this connects our abstract setting with much concrete classical analysis. Another semilattice S for which $R(S)$ is calculated is $S = \{1\} \cup X \cup \{0\}$ where $X \cup \{0\}$ is the one point compactification of a discrete infinite set X with multiplication $x^2 = x$ and all other products 0 and where 1 is an isolated identity. Then $R(S) = l^1(S)$ and $\|f\|_{R(S)} = |f(0)| + \sum_{x \in X} |f(x) - f(0)|$. Personally, I think that compact semilattices are an excellent test class to test the $R(S)$-theory; in general the authors do not tell us even for this comparatively narrow class whether or not $R(S)$ separates the points, let alone on the compact semitopological semigroups of type $\mathfrak{A}$; it would be distinctly desirable to have an answer to these questions. It is rather clear that $R(S)$ separates the points of a compact Lawson semilattice (i.e. one on which the morphisms into the min-interval separate the points). If for compact semilattices in general this were also true, then the $R(S)$-theory would afford a promising tool to attack the rather impermeable class of compact topological semilattices on which the morphisms into the min-interval do not separate the points; so far no tool whatsoever is known to attack these. What is certain is that semigroups with nilpotent elements cannot be treated with the $R(S)$-theory. Therefore, Dunkl and Ramirez seek and find a way to expand their theory of L^∞-representations. Recall that we first considered representations of S into the unit ball of commutative W^*-algebras, thus, in particular, of dual function algebras.

The authors enlarge the class of range algebras by considering dual Q-algebras, where a Q-algebra is simply the quotient algebra of some function algebra modulo a suitable clased ideal. Thus, a Q-representation is a weak-star continuous morphism $T: S \to A_1$ where A is a Q-algebra. The associated set of representative functions is called $RQ(S)$; a function $f \in C(S)$ belongs to $RQ(S)$ iff there is a Q-representation $T: S \to A_1$ and some $\omega \in A_*$ such that $f(s) = \langle T_s, \omega \rangle$ for all $s \in S$. Again it is shown that $RQ(S)$ is an involutive Banach algebra with a suitable norm whose dual $RQ(S)^*$ is a dual Q-algebra, and there is a universal Q-representation $\rho: S \to RQ(S)_1^*$ given by $\rho(s)(f) = f(s)$. The authors get much mileage out of the free Q-algebra generated by a set X which is explicitly constructed.

In what respects does the $RQ(S)$-theory improve the $R(S)$-theory? On the class $\mathfrak{U}$ which we considered before, the two theories essentially agree, since $R(S) = RQ(S)$ for $S \in \mathfrak{U}$ with $1 \in S$. On the other hand Dunkl and Ramirez demonstrate the greater generality of the Q-theory by calculating that $RQ(S)$ separates the points on any Rees quotient S of $(\mathbf{Z}^+)^X$ and $(\mathbf{R}^+)^n$; this takes care of the elusive nil-thread. It is remarkable that $RQ(S)$ measures the representability of S on Hilbert spaces. The authors study representations $\phi: S \to B(H)_1$ of S into the unit ball of the space of bounded operators on a Hilbert space equipped with the weak operator topology. If $L_\phi: C(S) \to B(H)$ is the linear map induced by ϕ and α is as above, then ϕ is called Q-bounded iff $L_\phi \circ \alpha$ is a linear contraction. They show that there exists for every commutative semitopological semigroup with 1 a Q-bounded representation ϕ such that the map $\phi_*: B(H)_* \to C(S)$ given by $\phi_*(\omega)(s) = \langle \phi(s), \omega \rangle$ satisfies $\operatorname{im} \phi_* = RQ(S)$. Moreover, $B(H)_*/\ker \phi_*$ and $RQ(S)$ are isometric via ϕ_*.

What we have indicated in this review, of course, cannot even circumscribe the scope of the monograph, let alone touch upon all subject matters treated or do justice to the respectable amount of technical work done, technical in the sense of traditional harmonic analysis. Considerable effort is made to discuss the harmonic analysis of discrete abelian semigroups which is one of the very early topics in the harmonic analysis of semigroups, initiated by Hewitt and Zuckerman in 1956. Not only is this a good testing ground, because one has a pretty firm grip on the situation, but one is highly motivated by the simple trick of discretification of a semitopological commutative semigroup, which surprising as this may sound, pays off in the topological theory by sometimes reducing a question to the discrete case. A wealth of material on Hilbert space representations and dilations covers a whole chapter.

The authors have to be credited with making every effort to make this book readable. Their reference and numerology system is excellent, since they give