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Embeddings and extensions in analysis, by J. H. Wells and L. R. Williams, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 84, Springer-Verlag, New York, Heidelberg, Berlin, 1975, vii + 108 pp., $14.40.

To embed a given space X into a standard, more distinguished space E, a mapping $\psi: X \to E$ must be exhibited, having certain prescribed properties. In this book E is a Banach space, e.g., a Hilbert space or, more generally, an L^p space, and ψ is an isometry into E. For $E = C[0, 1]$ the existence of such a $\psi: X \to E$, when X is an arbitrary separable metric space, is assured by a rather well-known result of Banach and Mazur. However, because of the strict constraints which are imposed on a metric when it is derived from an inner product, it is a priori clear that no such sweeping result is possible when E is a Hilbert space. Yet by virtue of its elevated position among Banach spaces one would wish to have a better knowledge of the precise conditions a metric space must satisfy to be embeddable in this particular standard space.

The first two chapters of the book provide a detailed answer to this question and to a number of related ones. (Here the contributions of I. Schoenberg, dating back to the late thirties, and those of Schoenberg and J. von Neumann of the early forties are truly impressive and rightly occupy a central position in the presentation.)

Without essentially altering the problem, one considers quasi-metric spaces (X, ρ) (i.e. pairs such that X is a set and ρ is a mapping of $X \times X$ into the nonnegative reals $\mathbf{R}^+$ with $\rho(s, t) = \rho(t, s)$ for all $s,\ t \in X$). A quasi-metric ρ is said to be of negative type if, for every finite subset $\{x_0, \ldots, x_n\}$ of X,

$$\sum_{j,k=0}^{n} \rho(x_j, x_k)^2 \xi_j \xi_k \leqslant 0 \quad \text{whenever} \sum_{j=0}^{n} \xi_j = 0.$$

It turns out that for a quasi-metric space to be (isometrically) embeddable into a sufficiently large Hilbert space it is necessary and sufficient that ρ be of negative type.

If F: $\mathbf{R}^+ \to \mathbf{R}^+$ is continuous, $F(0) = 0$, and $F \circ \rho$ is of negative type, then F itself is said to be of negative type and the collection of all such functions is denoted by $N(X)$. A related class of functions, called radial positive definite, consists of all continuous F: $\mathbf{R}^+ \to \mathbf{R}^+$ with the property that

$$\sum_{j,k=1}^{n} F(\rho(x_j, x_k))\xi_j \xi_k \geqslant 0$$

for all finite subsets $\{\xi_1, \ldots, \xi_n\}$ of the reals and all finite sets $\{x_1, \ldots, x_n\}$ in X; it is denoted by $RPD(X)$.

There is an intimate connection between $N(X)$ and $RPD(X)$ as

$$F \in N(X) \Leftrightarrow \exp(-\lambda F^2) \in RPD(X) \qquad (\lambda > 0).$$

Clearly then any information on $RPD(X)$ is interpretable in terms of embeddings, via $N(X)$, and vice versa. For example, by expressing $|t|^{2\alpha}$ in terms of the integral $\int_0^\infty (1 - \exp(-\lambda^2 t^2))\lambda^{-1-2\alpha}\, d\lambda$, it is possible to show that if $F \in N(X)$ and $0 < \alpha < 1$ then $\exp(-\lambda F^{2\alpha}) \in RPD(X)$ for all $\lambda > 0$. As a corollary it follows that the metric space which arises when, in Hilbert space, $\|x - y\|$ is replaced by $\|x - y\|^{\alpha}$ with $0 < \alpha \leqslant 1$ is embeddable in Hilbert space.

Having reached this stage in the development of the theory the emphasis shifts to a discussion of the analytical properties of $N(X)$ and $RPD(X)$ for special X, e.g., R^n and $L^p(\mu)$. Here, one of several interesting results is an integral representation for $N(R^n)$ stating that

$$F \in N(R^n) \Leftrightarrow F(t) = \left(\int_0^{\infty} \frac{1 - \Omega_n(tu)}{u^2} d(\alpha u)\right)^{1/2} \qquad (t \geqslant 0),$$

where

$$\Omega_n(t) = \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{t}\right)^{(n-2)/2} J_{(n-2)/2}(t)$$

(J_n denotes the Bessel function of order n), and α is a positive measure on $\mathbf{R}^+$ satisfying $\int_0^{\infty} d\alpha(u)/u^2 < \infty$. Culminating an exposition of the work of Jean Bretagnolle, Didier Dacunha Castelle and Jean Louis Krivine, along with more recent results of J. Kuelbs, the class $N(L^p)$ is described as follows. For $2 < p \leqslant \infty$ it consists of the zero function only; otherwise (i.e. for $0 < p \leqslant 2$),

$$F \in N(L^p) \Leftrightarrow F(t) = \left(\int_0^{\infty} \frac{1 - e^{-t^p u}}{u} d\alpha(u)\right)^{1/2} \qquad (t \geqslant 0),$$

with α positive measure on $\mathbf{R}^+$ such that $\int_1^{\infty} d\alpha(u)/u < \infty$.

A notable fact established already in the opening pages of the book is that for a metric space to be embeddable in a suitable Hilbert space it suffices that every finite subset be so embeddable, and this is later extended to embeddings into L^p spaces. (An adequate restatement of this fact in the latter case would read: if every finite subset of a given metric space X is isometric to a subset of an L^p space then there exists such a space which contains an isometric copy of the whole of X.) The proof, in both instances, is rather elaborate and, at least in the Hilbert space case, it would seem desirable that it be substantially simplified.

With the end of Chapter II the reader has to adapt to a somewhat abrupt transition to the other topic of the book, that of extensions. In general, if (X, d_1), (Y, d_2) are metric spaces the pair (X, Y) is said to have the contraction extension property if every nonexpansive mapping from a subset S of X into Y extends to such a mapping from the whole of X into Y; and similarly, one defines the corresponding properties for isometries and Lipschitz-Hölder maps, the latter involving the parameters k, α ($0 \leqslant k, 0 < \alpha \leqslant 1$) in the inequality $d_2(Tx_1, Tx_2) \leqslant k d_1(x_1, x_2)^{\alpha}$. Numerous variants arise for different choices of X and Y, different domains and ranges and different types of maps.

Apart from a digression in Chapter IV, some fifty pages are devoted to a variety of such problems with a predominance of those in which both X and Y are Banach spaces. The discussion begins with a restatement of the contraction extension property in terms of the familiar Kirszbraun property (K).

(X, Y) is said to have property (K), if, for any fixed set I, and any pair of families $\{B(x_i, r_i): i \in I\}$, $\{B(y_i, r_i): i \in I\}$ of closed balls in X and Y, respectively, such that $d_2(y_i, y_j) \leqslant d_1(x_i, x_j)$ $(i,j \in I)$,

$$\bigcap B(x_i, r_i) \neq \varnothing \Rightarrow \bigcap B(y_i, r_i) \neq \varnothing.$$

Property (K) is then shown to be equivalent to the contraction extension property. This fact is helpful in showing that for a Hilbert space H, the pair (H, H) has the extension property for Lipschitz-Hölder maps. (This result is generalized to pairs (L^p, L^q) in the closing pages of the book.) Moreover, within the class of strictly convex Banach spaces no other pair (X, X) has this property. A similar result, due to S. Schönbeck, holds for pairs (X, Y) where Y is strictly convex and $\dim Y \geqslant 2$, though the proof of this fact is considerably more involved. Without strict convexity of Y the problem is, in general, rather difficult and partial solutions are, therefore, of interest. One of these due to B. Grünbaum (for $\dim X = 2$) and to S. Schönbeck, states that if X is a separable conjugate Banach space, then for (X, X) to have the contraction extension property it must be a Hilbert space or have the binary intersection property for closed balls. Along with property (K) and the above mentioned property, other intersection theorems for families of closed balls are known to be useful when dealing with the extension of contractions, and some of these are presented in that context. Briefly touched upon are the investigations of D. de Figueiredo and L. Karlovitz into the existence of contractive retractions over a closed convex subset of a Banach space, as well as those of F. Valentine on the contraction extension property of (X, X) when X is an n-sphere. In a departure from the main theme several other loosely connected topics are dealt with. Among these are the extension problem for uniformly continuous mappings, and, in a different direction altogether, a packing problem for the unit ball in L^p.

For a slim volume, about a hundred pages long, the amount of material covered is considerable, and while the authors may have omitted some topics which are relevant to the subject matter, and included others which are less so, the balance seems satisfactory. The writing is exceedingly clear and the pace easy to keep up with.

This book should help to arouse a more widespread interest in an area in which the interplay between geometry and analysis is both fruitful and pleasing. As such, it is most welcome.

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A geometric approach to homology theory, by S. Buoncristiano, C. P. Rourke, and B. J. Sanderson, London Mathematical Society Lecture Note Series, no. 18, Cambridge Univ. Press, New York and London, 1976, 149 pp., $10.95.