$(1-x)$ and $(1-y)$ are replaced by $(1-x)^\alpha$ and $(1-y)^\alpha$ $(\alpha \neq 1)$: the limit of the generalized information function when α tends to 1 is the Shannon's information function.

In Chapter VII further generalizations of Rényi's entropy are introduced containing two parameters α, β: if $\beta = 1$ they reduce to Rényi's entropy.

The book of J. Aczél and Z. Daróczy represents the summing-up of a long series of fruitful researches: one has the impression that they have so thoroughly explored the field, that there is little chance for the discovery of really new properties of Shannon's entropy and eventually Rényi's entropy; perhaps this outstanding achievement, discouraging further efforts on the same line, will now stimulate explorations of neighbouring fields, taking account of all the aspects of information out of the scope of the classical theory.

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Differentiation of integrals in R^n, by Miguel de Guzmán, Lecture Notes in Mathematics, no. 481, Springer-Verlag, Berlin, Heidelberg, New York, 1975, xi + 225 pp., \$9.50.

Professor de Guzmán's book concerns itself with material which has come, in recent years, to play a fundamental role in the theories of real and complex analysis, Fourier analysis, and partial differential equations. Maximal functions, covering lemmas and differentiation of integrals seem to be at the core of the modern theory of singular integrals, Littlewood-Paley theory, and H^p spaces, as well as many other areas of great interest.

The starting point of the theory is the consideration of the following simple result:

Given $f \in L^1(R^n)$, we have

$$\lim_{r\to 0} \frac{1}{|B(x;r)|}\int_{B(x;r)} f(y)\,dy = f(x)$$

for a.e. $x \in R^n$. (Here $B(x;r)$ is the ball centered at x of radius r, and $|B(x;r)|$ is its Lebesgue measure.) This result, known as Lebesgue's theorem on the differentiation of the integral, is, however, just the beginning of the theory. For, in order to give their proof of this result, Hardy and Littlewood introduced the maximal operator, M, given by

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x;r)|}\int_{B(x;r)} |f(y)|\,dy, \qquad f \in L^p(R^n), \qquad 1 \leqslant p \leqslant \infty.$$

This maximal operator, which is of fundamental importance in many areas, turns out to be bounded on $L^p(R^n)$ when $p > 1$, and majorizes some of the most important operators in Fourier analysis. For example, the process of taking Cesàro means of Fourier series or Poisson integrals of functions can be understood via the Hardy-Littlewood maximal operator. This operator, M, in turn is best understood in terms of a certain covering lemma. Roughly speaking, the covering lemma says that if a set $E \subset R^n$ is covered by a union of overlapping balls, then a disjoint subcollection of these balls may be chosen in such a way that the disjoint subcollection has total measure at least equal to a fixed fraction of the measure of E. The idea of a covering lemma is a crucial one, and there are a huge number of variants of the lemma above in the literature.

Thus, the introduction of the basic problem of differentiation of the integral by Lebesgue and the subsequent realization that a solution of the problem could be found which had a profound relationship with other important problems of analysis provided the foundation upon which much of the later theory of real and complex analysis is built.

A particularly striking example of this enormous impact of maximal functions on problems in Fourier analysis and partial differential equations is the theory of singular integral operators. A. P. Calderón and A. Zygmund, in their fundamental study of singular integral operators made use of a certain decomposition of a function into large and small parts, and this decomposition was closely related to the maximal function in a beautifully subtle way. In fact, Calderón and Zygmund obtained a new proof of the appropriate L^1 inequality for the maximal operator using their decomposition. In the following decades, the Calderón-Zygmund methods were ingeniously applied to solve a great number of problems in analysis, many relating directly back to the source–differentiation theory. A good example of this is E. M. Stein's characterization of the functions f on the unit cube Q in R^n for which $M(f) \in L^1(Q)$. Stein showed that $M(f) \in L^1(Q)$ if and only if $\int_Q |f|(\log^+|f| + 1)\,dx < \infty$.

Another example is the important work of F. John and L. Nirenberg which introduced the class BMO of functions of bounded mean oscillation, which plays a crucial role in the C. Fefferman-Stein theory of H^p spaces. Finally, we would like to mention our last example here of the application of the Calderón-Zygmund decomposition to areas of Fourier analysis. This is the idea of the Burkholder-Gundy inequalities: Suppose that we know that a linear operator S is bounded on $L^p(R^n)$ and we have another operator T on $L^p(R^n)$ which we wish to prove is bounded. If we can show that for small enough $\gamma > 0$ and all $\alpha > 0$ we have

$$m\{|T(f)| > 2\alpha, |S(f)| < \gamma\alpha\} \leqslant C\gamma m\{|T(f)| > \alpha\},$$

then it will follow easily that T is also bounded on $L^p(R^n)$. Burkholder and Gundy were able to show that such an inequality is valid when T is the Hilbert transform and S is the Hardy-Littlewood maximal function. The proof of this inequality makes use of the Calderón-Zygmund decomposition and it is a vivid demonstration of the fact that the Hilbert transform is controlled by the maximal operator. Such inequalities have been used to establish weight norm inequalities and in order to interpolate between the spaces H^1 and L^p, this interpolation being a basic idea in the C. Fefferman-Stein theory of H^p spaces. So it is to say the very least that the theory of singular integrals, which has such deep roots in differentiation theory, has had an impressive variety of applications.

Somewhat later than the results of Calderón and Zygmund on the boundedness of singular integral operators a major contribution to Fourier analysis was given by E. M. Stein in his famous work on limits of sequences of operators. As mentioned above, it had been known to analysts for a long time that in order to establish the existence of pointwise limits of a sequence of operators (such as the averages of functions over balls with a fixed center, the Hilbert transform, and the partial sums of a Fourier series) it was enough to prove a maximal inequality. That is, if one wanted to establish that for $f \in L^1(0, 2\pi)$ and a sequence T_n of bounded linear operators on $L^1(0, 2\pi)$,

$$\lim_{n\to\infty} T_n(f)(\theta) \text{ exists for a.e. } \theta \in (0, 2\pi),$$

then one ought to try to prove that

$$m\{T^*(f) > \alpha\} \leqslant \frac{C}{\alpha}\|f\|_1,$$

where

$$T^*(f)(\theta) = \sup_{n \geqslant 1} |T_n(f)(\theta)|$$

is the maximal operator. Stein proved the amazing result that for a very large class of operators, if the limits in question exist a.e., then the maximal inequality above *must* hold. This theorem has had a great number of important consequences, one of which should be mentioned now. This result is that a differentiation theorem is equivalent to a maximal theorem. To illustrate, consider the question of whether the collection, $\mathfrak{R}$, of all rectangles in R^2 with sides parallel to the coordinate axes differentiates the integrals of functions in $L^1(R^2)$. That is, is it true that

$$\lim_{\substack{R \in \mathfrak{R} \\ x \in R, \operatorname{diam}(R) \to 0}} \frac{1}{|R|} \int_R f(y)\, dy = f(x) \quad \text{for a.e. } x \in R^2$$

whenever $f \in L^1(R^2)$? It follows at once from Stein's theorem that the answer to the preceding question is "no" because the simplest examples show that the maximal operator

$$M_{\mathfrak{R}}(f)(x) = \sup_{x \in R; R \in \mathfrak{R}} \frac{1}{|R|} \int_R |f(y)|\, dy$$

does not satisfy an inequality of the form

$$m\{M_{\mathfrak{R}}(f) > \alpha\} \leqslant \frac{C}{\alpha}\|f\|_1.$$

In fact, when his result first appeared, Stein applied it to eliminate many difficult open problems in Fourier analysis. Since then, other applications have appeared, and one of the more recent is discussed below. It should be pointed out that, despite the emphasis here on applications, the result is, in itself, one of the most basic principles of real and complex analysis.

Connected with the theorem of Stein is another idea related to the theory of differentiation due to Charles Fefferman. This is his beautiful solution to the so-called "disk conjecture", where he provided a negative answer to the following question: Is the operator T_D defined by $T_D(f)\hat{}\,(\xi) = \chi_D(\xi) \cdot \hat{f}(\xi)$ bounded on $L^p(R^2)$ for any $p \neq 2$? (Here χ_D is the characteristic function of the unit disk in the plane and $\hat{f}$ is the Fourier transform of f.) C. Fefferman was able to show the fallacy of the conjecture by using the solution to the Kakeya needle problem, which is itself intimately connected to the differentiability of integrals. Finally, if we combine these striking results of C. Fefferman (on T_D) and E. M. Stein (on limits of sequences of operators) it is not difficult to see that for $p < 2$ there exists a function $f(\theta_1, \theta_2) \in L^p(T^2)$ with double Fourier series

$$f \sim \sum_{m,n=-\infty}^{+\infty} a_{mn} e^{i(m\theta_1 + n\theta_2)}$$

so that

$$\lim_{R\to\infty} \sum_{m^2+n^2 \leqslant R^2} a_{mn} e^{i(m\theta_1 + n\theta_2)}$$

fails to exist on a set of positive Lebesgue measure on the torus. Put in plain terms, on $L^p(T^2)$, $p < 2$ the disk partial sums of the Fourier series need not converge a.e. This remarkable result stands in sharp contrast to the famous one-dimensional result of Carleson-Hunt which says that the Fourier series of a function in $L^p(0, 2\pi)$ converges a.e. as soon as $p > 1$.

The results above represent some great highlights in the history of analysis in the present century. Though at first glance many of them would seem to have nothing at all to do with the theory of differentiation, it has turned out that each of these problems and their solutions have had roots reaching deep into this theory. Furthermore the results here hardly begin to exhaust the list of important theorems of analysis relating to differentiation of integrals. And we do not wish to leave the impression that the interplay between analysis and differentiation is now completely understood, and will be of no use in the mathematics of tomorrow. On the contrary, the field is now blossoming faster and becoming more important than ever. Two good examples of this development are the recent work of A. Cordoba, which connects the study of the maximal operator with respect to general differentiation bases with the covering properties of these bases, and the recent results of E. M. Stein, A. Nagel, N. Rivière, and S. Wainger dealing with maximal operators and singular integrals along lower dimensional varieties in R^n. Because of these developments as well as others it seems assured that differentiation theory will take its place in the future study of analysis.

At this point, some more specific remarks on the nature of the author's book are in order. The book is not primarily intended to treat the applications of differentiation theory to real and complex analysis. Rather, for the most part, the author tries to achieve great depth in treating the "pure" differentiation theory itself. He therefore provides a background for the material discussed here rather than this material proper. De Guzmán's book contains the basic Lebesgue theorem on differentiation of the integral and the Hardy-Littlewood maximal theorem along with a great many variants of these theorems, proven by the use of covering lemmas. The variants of the Vitali lemma which the author treats are also quite numerous. It is extremely commendable that the Calderón-Zygmund decomposition is proven, and the disk multiplier problem is mentioned, with a few words about C. Fefferman's solution to the problem. In addition, De Guzmán includes a careful treatment of differentiation theory with respect to two other extremely crucial differentiation bases besides the class of balls (or what is essentially the same thing, cubes). These are the bases of all rectangles in R^n with sides parallel to the coordinate axes, and the larger class of all rectangles in R^n with arbitrary orientation. The relationship between covering lemmas, maximal theorems, and differentiation theorems is also discussed. These important topics, as well as many others make the book's content worthwhile for the experts of this subject or for students who would like to learn these areas, and then branch out by studying the important applications.

De Guzmán's book is carefully written, and the style makes for easy and enjoyable reading. His work is a significant contribution to the field which should be welcomed by all concerned with this beautiful area of mathematics.

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Invariants for real-generated uniform topological and algebraic categories, by Kevin A. Broughan, Lecture Notes in Mathematics, No. 491, Springer-Verlag, Berlin, Heidelberg, New York, 1975, x + 197 pp., \$8.20.

The literature on relations between the dimension of a metrizable space X and the existence of metrics on X having convenient special properties is rather extensive (see Nagata's book [5] for the only good exposition) and contains two really successful theorems. First, Hausdorff formalized the idea of estimating the measure of a set A in t-dimensional space by covering it with finitely many ε_i-spheres and taking their measures to be ε_i^t. It turns out (L. Pontrjagin and L. Schnirelmann, 1932; E. Szpilrajn, 1937; book [5, pp. 112–116]) that the dimension of separable metrizable A is the infimum of the real numbers t such that for some metric on A, the t-dimensional measure is 0. The second theorem is P. Ostrand's [6] (improving results of J. de Groot, 1957, and J. I. Nagata, 1958; book [5, pp. 137–154]): metrizable X has covering dimension $\leqslant n$ if and only if it has a metric in which, for all ε, given