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BOOK REVIEWS

Large infinitary languages, model theory, by M. A. Dickmann, Studies in Logic and the Foundations of Mathematics, Vol. 83, North Holland/American Elsevier, Amsterdam and Oxford/New York, xv + 464 pp., $39.95.

This book is a comprehensive compilation of model theoretic results concerning the languages $L_{\kappa\lambda}$. Most of the results have been previously published over the last 15 or 20 years, but they are carefully organized and rounded out in the book. Although containing a large amount of highly technical material, the book is surprisingly readable due to the informal remarks which keep the reader informed as to the objectives of and the methods used in the detailed presentation.

The languages, as well as the problems considered, are strongly involved with cardinality consideration. As defined by Tarski in a 1958 paper (with $\kappa = \lambda$), $L_{\kappa\lambda}$ is a first order language making use of the same models and atomic formulas as ordinary predicate logic (which can be identified with $L_{\omega\omega}$). However, in building up formulas, conjunctions of sets of formulas of power less than κ and quantifications over sets of variables of power less than λ are allowed. If κ is bigger than ω, this, of course, allows formulas which are not "writable" in the sense of being finite strings of symbols. Also, in the book, no notions of recursiveness, admissibility, or even definability are applied to the formulas or sets of formulas considered. Nevertheless, the purely model theoretic notions developed for $L_{\omega\omega}$ are easily applied to these languages and the results obtained turn out to be quite different.

The reader is expected to have a fairly broad background in logic and set theory. Some specialized topics in set theory that he is less likely to be familiar with are taken up in Chapter 0. One section takes up filters, ultraproducts and trees; another takes up the transition from countably additive real-valued measures on all subsets of a set to the existence of κ-additive two-valued measures on a cardinal κ; and still another takes up the partition calculus and the relation $m \to (n)^r_\kappa$. Of even greater importance are the sections on cardinal arithmetic and the Mahlo hierarchies of inaccessible cardinals. The interested reader should work out for himself, at each stage of the string of definitions, some further facts concerning the size of the cardinals being considered. For example, note how much bigger the gap is between the second and third inaccessible (if they exist) than between the first and second. Only then will he be equipped to judge for himself the importance of theorems in the remainder of the book concerning cardinals much larger than these and whose existence is much more conjectural.

Chapter 1 introduces the languages, their truth definition, and other basic model theoretic concepts. Some of the expressive power of the languages is revealed in examples. It is interesting to note that some concepts that are most naturally expressed in second order language (such as the concept of a well-ordered set or a simple group) can also be expressed by a denumerable first order formula. Other second order conditions (such as a set being the power set of another set and various completeness conditions with unrestricted cardinality) cannot be expressed in any of the languages $L_{\kappa\lambda}$. Many of the model theoretic results in the book, or the examples used in their proof, provide further information concerning the expressive power of the languages.

Craig's interpolation theorem and Beth's definability theorem extend to $L_{\omega_1\omega}$, but not to larger languages as shown by examples in Chapter 2. Certain preservation theorems extend to $L_{\theta\theta}$ for θ an inaccessible cardinal, but not to other cardinals as shown in Chapter 5. The compactness property, so widely used in ordinary logic, fails for accessible cardinals and all reasonably sized inaccessible cardinals. Chapter 3 gives these results and explores a large number of conditions equivalent to or implied by the compactness theorem for $L_{\theta\theta}$ if such a rare and necessarily very large cardinal could possibly exist.

Much of Chapters 3 and 4 and part of Chapter 5 is devoted to the Löwenheim-Skolem theorem for the languages. In the downward direction, finding an $L_{\kappa\lambda}$-elementary subsystem of a smaller cardinality, the ordinary techniques work except that the presence of the infinite quantifiers imposes a much more stringent condition on the cardinality of the subsystem. In the upward direction, finding an $L_{\kappa\lambda}$-elementary extension of a given larger cardinality, the ordinary techniques, which depend on some form of compactness, fail. Let $h(L_{\kappa\lambda})$ be the least cardinal α such that, for any sentence σ of $L_{\kappa\lambda}$, if σ has a model of power at least α, then σ has models of arbitrarily large powers. Most of Chapter 4 is devoted to the evaluation of $h(L_{\kappa\omega})$ which turns out to be equal to a critical cardinal defined by Morely in connection with omitting types in ordinary logic. Thus this chapter, together with the background on types of elements, saturated structures, and indiscernibles given in Chapter 2, will be of interest to the reader exclusively interested in the model theory of $L_{\omega\omega}$. Concerning the general case or even $h(L_{\omega_1\omega_1})$, the known results, which are summarized in the first section of Chapter 5, are even less definitive and involve very large inaccessible cardinals.

Most of these results give little information when it comes to applications. Suppose we wish to apply $L_{\kappa\lambda}$ to a particular theory T of a class of models K. The compactness theorem, for example, while failing in general, may apply to the theory T. Stronger versions of the downward as well as the upward Löwenheim-Skolem theorem may apply in the particular case in question. All these things would come true, for example, if the predicates of T were all unary. Thus the concepts of finitary model theory do not fit the infinitary situation as well. New concepts seem to be needed in order to formulate more powerful theorems for infinitary logic. One set of concepts, however, that does seem to fit into the infinitary situation well is the back and forth method of extensions of partial isomorphisms. In the last part of Chapter 5, these methods are developed in a very general form and some applications are given.

With its index, bibliography, and numerous problems, the book should provide a useful guide for experts or prospective experts in this specialized field of logic.

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