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A NEW CHARACTERIZATION OF PLANAR GRAPHS

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We consider graphs on a finite set of vertices. In addition the graphs are undirected, although, for the purposes of the characterization, we will need to give each edge a direction. We use the notation EG to denote the set of edges of the graph G, and VG for the corresponding vertex set. A graph is defined to be planar if and only if it can be embedded in the plane so that any two edges intersect at a common end-vertex or not at all.

Our characterization of planar graphs is given by the following theorem. For other characterizations, see [1].

THEOREM 1. *A graph is nonplanar if and only if it contains a maximal, strict, compact, odd ring.*

We now explain the terms introduced in the characterization.

Let S be a collection of n circuits in a graph G and suppose that the edges of G may be directed so that each circuit of S is a directed circuit. We say that, for $n \geq 3$, S is a *ring* if (i) the circuits of S can be labelled $C_0, C_1, \ldots, C_{n-1}$ so that $EC_i \cap EC_j \neq \emptyset$ if and only if $i = j$, $i \equiv j + 1 \bmod n$ or $i \equiv j - 1 \bmod n$, and (ii) no edge of G belongs to more than two circuits of S.

(We note that (i) implies (ii) except when $n = 3$.)

The ring S is said to be *strict* if $|VC_i \cap VC_j| \leq 1$ whenever $EC_i \cap EC_j = \emptyset$; it is said to be *maximal* if there does not exist a ring $\{C'_0, C'_1, \ldots, C'_{m-1}\}$ in G such that $\bigcup_{k=0}^{m-1} EC'_k \subseteq \bigcup_{l=0}^{n-1} EC_l$ and $m > n$. If n is odd, then S is said to be *odd*. Finally, S is *compact* if there is no ring $\{C''_0, C''_1, \ldots, C''_{n-1}\}$ such that $\bigcup_{k=0}^{n} EC''_k \subset \bigcup_{l=0}^{n} EC_l$.

The characterization is proved by purely combinatorial means which are motivated by topological considerations. We set out the main steps of the proof in the sequence of lemmas below.

LEMMA 1. *Every nonplanar graph contains a maximal, strict, compact, odd ring.*

The proof of this lemma follows from [2], where a maximal, odd ring is constructed in every nonplanar graph.

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LEMMA 2. *If G contains a maximal ring with just three circuits then G is nonplanar.*

As a result of this lemma we are free to concentrate on odd rings with five or more circuits. We then show that no two consecutive circuits in our ring may have more than one nontrivial path in common.

LEMMA 3. *If G contains a maximal, strict, compact, odd ring with more than three circuits in which some pair of circuits C_i, C_{i+1} have more than one nontrivial path in common, then G is nonplanar.*

The final lemma concludes the proof of the characterization.

LEMMA 4. *If G contains a maximal, strict, compact, odd ring with more than three circuits in which any pair of circuits C_i, C_{i+1} have precisely one nonempty path in common, then G is nonplanar.*

REFERENCES

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