assert their inspiration lies in physics, few of them face up to the fact that physics is an experimental science so that theories are of maximal use confronting numbers experimentalists observe in the laboratory. For a long while, mathematicians have restricted their interest in numbers to statements such as: there exist no nonvanishing vector fields on spheres of even dimension or that the set of isomorphism classes of k-dimensional vector bundles over a paracompact space B has a natural bijective correspondence with the set of homotopy classes of mappings of B into the Grassmann manifold of k-dimensional subspaces of an infinite dimensional space. We have passed the art of computation along to computerologists–selling both ourselves and the world out.

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Vector measures and control systems, by Igor Kluvanek and Greg Knowles, North-Holland Mathematics Studies, vol. 20, North · Holland / American Elsevier, Amsterdam/New York, 1975, ix + 180 pp., $13.50.

The theory of vector measures has been under increasingly heavy study for the last decade. By the early seventies coherent bodies of knowledge had solidified in the areas of vector measure theory that grew from either the Orlicz-Pettis theorem or the Dunford-Pettis Radon-Nikodým theorem for the Bochner integral. But as late as 1974 the range of a vector measure was still an object of some mystery.

At that time the two main theorems about the range of a vector measure were Liapunov's convexity theorem (the range of a nonatomic vector measure with values in a finite dimensional space is compact and convex) and the Bartle-Dunford-Schwartz theorems (a vector measure with values in a Banach space has a relatively weakly compact range and is absolutely continuous with respect to a scalar measure). The infinite dimensional version of Liapunov's theorem remained a particular enigma; Liapunov had shown, by example, that his convexity theorem failed for vector measures with values in the sequence spaces l_p ($1 \leqslant p \leqslant \infty$). The very scope of Liapunov's example served to block serious research into the infinite dimensional version of Liapunov's convexity theorem. This, in turn, held up the understanding of the bang-bang principle for control systems with infinitely many degrees of freedom (e.g. a control system governed by a partial differential equation).

Also in the early seventies it became clear that a sharpened form of the Bartle-Dunford-Schwartz theorem was needed. It was realized that the range

of a vector measure is distinguished by properties far stronger that relative weak compactness. The unit ball of any reflexive quotient of $C([0, 1])$ is the range of a vector measure, but the unit ball of three-dimensional l_1 is not. Some work of Bolker's in the finite dimensional case indicated that the range of a vector measure with values in a Banach space should be characterized by its geometry. But, other than what can be deduced from the Kreĭn-Mil'man theorem, little was known about the geometry of the range of a vector measure.

The years 1973–1975 saw marked progress in the study of the range of a vector measure, progress to which the authors of this book have contributed in no small way. This welcome contribution to the literature is their report on the current state of the knowledge about the range of a vector measure and its applications to control systems.

I think the highlight of the book is the author's presentation of the infinite dimensional version of Liapunov's theorem. By Liapunov's example, non-atomicity must have a stronger meaning in the finite dimensional case than it has in the infinite dimensional case. Knowles in 1974, partially aided by a theorem of Kingman and Robertson, was able to isolate the essential kernel of the meaning of nonatomicity in Liapunov's theorem. In so doing he made Liapunov's theorem into one of the charms of measure theory. I cannot resist trying to communicate the basic idea of Knowles' approach.

Let X be a Banach space, Σ be a σ-field and $m: \Sigma \to X$ be a vector measure. Call m Liapunov if for each $A \in \Sigma$ the set $\{m(E \cap A): E \in \Sigma\}$ is weakly compact and convex. By the Bartle-Dunford-Schwartz theorem there is a finite positive measure μ on Σ with exactly the same null sets as m. For each $E \in \Sigma$ of positive μ-measure let $L_\infty(\mu, E)$ be the subspace of $L_\infty(\mu, E)$ consisting of functions vanishing off E. Note that if X is finite dimensional and m is nonatomic then the operator $f \to \int_E f \, dm, f \in L_\infty(\mu, E)$, is many-to-one on each $L_\infty(\mu, E)$ because each $L_\infty(\mu, E)$ is infinite dimensional. This is the kernel of nonatomicity in the finite dimensional case. For general Banach spaces X, Knowles proved that m is Liapunov if and only if this operator is many-to-one on each $L_\infty(\mu, E)$. Some of the charm of Knowles' theorem derives from its elegant proof which is based in part on Lindenstrauss' well-known proof of the finite dimensional version of Liapunov's theorem.

Here is an easy consequence of Knowles' theorem: A vector measure is Liapunov if and only if its range is mid-point convex, a fact originally proved in the finite dimensional case by Halmos.

After giving a thorough discussion of Liapunov's theorem, the authors relate Liapunov vector measures to the bang-bang principle of control theory. This is "an attempt to extend the approach and results concerning the control of systems with a finite number of degrees of freedom to systems governed by partial differential equations." A number of concrete examples are given including the establishment of the bang-bang principle for certain systems governed by the wave equation, the heat equation and the diffusion equation. There is also an illuminating discussion that reveals that St. Venant's principle

in linear elasticity theory implies that the relation between the deformations (or stresses) and forces causing them is expressible as integration with respect to a Liapunov vector measure. This means that St. Venant's principle implies the bang-bang principle.

The uniqueness of controls for systems governed by partial differential equations is related to the extreme points of the attainable set which, in turn, is often related to the range of a vector measure. Thus, once the bang-bang principle is understood, it is natural to examine the extremal structure of the range of a vector measure. The authors give a tightly organized presentation of this subject. The main theorems are as follows: Every extreme point of the closed convex hull of the range of a vector measure belongs to the range of the measure (Liapunov). Every extreme point of the range is a denting point (Anantharaman). Every exposed point of the range of a vector measure is strongly exposed (Anantharaman). Also included here is Anantharaman's proof of Rybakov's theorem. This proof is a beautiful melding of Banach space theory and measure theory to the profit of both fields.

Next the authors consider the problem of characterizing those subsets of Banach spaces that are ranges of vector measures. As the authors state, "The problem to construct for a given set K a vector measure (whose range is K) is unreasonably ambitious. More tractable is the following one. If K is a convex set, find a vector measure m such that K is the closed convex hull of the range of m." They show that this is not much of a compromise by showing that the closed convex hull of the range of a vector measure is itself the range of a (Liapunov) vector measure. The central theorem is due to Kluvanek and is based on Choquet's conical measures. This theorem reveals that a set K is the closed convex hull of the range of a vector measure if and only if K is a zonoform. This is the definitive extension of Bolker's earlier (finite dimensional) work. This is then combined with some theorems of Choquet's and Herz's to relate ranges of vector measures to weakly compact convex symmetric sets with negative-definite support functionals. As the authors freely admit, these characterizations are, at present, possibly a bit too abstract to be of immediate use in vector measure theory or control theory. More must be known about zonoforms. On the other hand, they do allow vector measure theory to be applied to the theory of zonoforms. Thus, for instance, every exposed point of a zonoform is strongly exposed. Also these results may be of use in the problem of constructing a control system with a given attainable set.

The book closes with a treatment of optimal control for systems with infinitely many degrees of freedom steered by a sequence of independently operating controls. The work here is "a contribution to the programme of extending (the approach of Hermes and LaSalle, *Functional analysis and time optimal control*, Academic Press, New York, 1969) to the infinite dimensional situations."

One misimpression should be rectified here. Although I have stated the theorems above for Banach spaces only, this book is executed in the context

of measures with values in locally convex spaces. Sometimes the theory becomes a bit more complicated than I have indicated above, but Kluvanek's notion of a closed vector measure is used to hold complications to a minimum. Every vector measure with values in a metrizable locally convex space is closed; thus closed vector measures seem to be the natural generalizations of Banach space-valued measures to the locally convex context. I hasten to add that this generalization is not an idle extension that searches for generality for the sake of generality. Indeed a functional analytic approach to many concrete problems of control systems is impossible within the context of Banach spaces.

There are several reasons to be thankful for this book. In addition to bridging the gap between pure and applied mathematics, it is the definitive work on the range of a vector measure. It reads easily and its literature surveys (which appear at the end of each chapter) are chock-full of tidbits of information that are useful to the student, scholar or researcher. It is a book worth having and using.

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