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NONLINEAR ERGODIC THEOREMS

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In two recent notes ([1], [2]), J-B. Baillon proved the first ergodic theorems for nonlinear mappings in Hilbert space. We simplify the argument here and obtain an extension of Baillon's theorems from the usual Cesàro means of ergodic theory to general averaging processes $A_n = \Sigma_{k=0}^{\infty} a_{n,k} T^k$ $(0 \leqslant a_{n,k}, \Sigma_{k \geqslant 0} a_{n,k} = 1)$.

THEOREM 1. *Let H be a Hilbert space, C a closed bounded convex subset of H, T a nonexpansive self map of C. Suppose that as $n \longrightarrow \infty$, $a_{n,k} \longrightarrow 0$ for each k, and $\gamma_n = \Sigma_{k=0}^{\infty}(a_{n,k+1} - a_{n,k})^+ \longrightarrow 0$. Then for each x in C, $A_n x = \Sigma_{k=0}^{\infty} a_{n,k} T^k x$ converges weakly to a fixed point of T.*

The proof of Theorem 1 depends upon an extension of Opial's lemma [3].

LEMMA 1. *Let $\{x_k\}$ and $\{y_k\}$ be two sequences in H, F a nonempty subset of H, C_m the convex closure of $\bigcup_{j \geqslant m} \{x_j\}$. Suppose that*

(a) *For each f in F, $|x_j - f|^2 \longrightarrow p(f) < +\infty$;*

(b) $\operatorname{dist}(y_k, C_m) \longrightarrow 0$ *as $k \longrightarrow \infty$ for each m;*

(c) *Any weak limit of an infinite subsequence of $\{y_k\}$ lies in F.*

Then y_k converges weakly to a point of F.

PROOF OF LEMMA 1. Since $\{y_k\}$ is bounded, it suffices to show that if f and g in F are weak limits of infinite subsequences of $\{y_k\}$, then $f = g$. For each j,

$$|x_j - f|^2 = |x_j - g|^2 + |g - f|^2 + 2(x_j - g, g - f).$$

For a given $\epsilon > 0$, there exists $m(\epsilon)$ such that for $j \geqslant m(\epsilon)$,

$$|p(g) - |x_j - g|^2| < \epsilon; \qquad |p(f) - |x_j - f|^2| < \epsilon.$$

Let K_ϵ be the convex set of all u such that

$$|2(u - g, g - f) + p(g) - p(f) + |g - f|^2| \leqslant 2\epsilon.$$

Since K_ϵ contains $\bigcup_{j \geqslant m(\epsilon)} \{x_j\}$, it contains $C_{m(\epsilon)}$. There exists k_ϵ such that for $k \geqslant k_\epsilon$ we can find u_k in $C_{m(\epsilon)}$ such that $|y_k - u_k| \leqslant \epsilon$. For $k \geqslant k_\epsilon$, it follows that

$$|2(y_k - g, g - f) + p(g) - p(f) + |g - f|^2| \leqslant 2\epsilon + 2\epsilon|g - f|.$$

Consider an infinite subsequence $\{y_{k_s}\}$ for which $(y_{k_s} - g, g - f) \longrightarrow 0$. In the limit

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$$|p(g) - p(f) + |g - f|^2| \leqslant 2\epsilon + 2\epsilon|g - f|.$$

Since $\epsilon > 0$ is arbitrary, it follows that $p(g) + |g - f|^2 = p(f)$. By symmetry, $p(f) + |g - f|^2 = p(g)$. Hence, $|f - g|^2 = 0, f = g$. Q.E.D.

PROOF OF THEOREM 1. We apply Lemma 1 with F the fixed point set of T in C, $x_k = T^k x$, $y_n = \Sigma_{k \geqslant 0} a_{n,k} x_k$. Since $|x_j - f|^2$ decreases with j, it converges to $p(f) < +\infty$. Since $a_{n,k} \longrightarrow 0$ as $n \longrightarrow +\infty$, $\operatorname{dist}(y_n, C_m) \longrightarrow 0$ $(n \longrightarrow +\infty)$. To show that (c) holds, it suffices to prove that $|y_n - Ty_n| \longrightarrow 0$ as $n \longrightarrow +\infty$. For any u in H,

$$|y_n - u|^2 = \left|\sum_{k \geqslant 0} a_{n,k}(x_k - u)\right|^2 = \sum_{j,k \geqslant 0} a_{n,j} a_{n,k}(x_j - u, x_k - u).$$

Since $2(x_j - u, x_k - u) = |x_j - u|^2 + |x_k - u|^2 - |x_j - x_k|^2$,

$$2|y_n - u|^2 = 2\sum_{k \geqslant 0} a_{n,k}|x_k - u|^2 - r_n,$$

where $r_n = \Sigma_{j,k \geqslant 0} a_{n,j} a_{n,k} |x_j - x_k|^2$. If we choose $u = y_n$, then

$$r_n = 2\sum_{k \geqslant 0} a_{n,k}|x_k - y_n|^2.$$

If we set $u = Ty_n$, we find that

$$\begin{aligned} 2|y_n - Ty_n|^2 &= 2a_{n,0}|x - Ty_n|^2 + 2\sum_{k \geqslant 1} a_{n,k}|Tx_{k-1} - Ty_n|^2 - r_n \\ &\leqslant 2a_{n,0}|x - Ty_n|^2 + 2\sum_{k \geqslant 1} a_{n,k}|x_{k-1} - y_n|^2 - 2\sum_{k \geqslant 0} a_{n,k}|x_k - y_n|^2 \\ &\leqslant 2a_{n,0}|x - Ty_n|^2 + 2\sum_{k \geqslant 0} \{a_{n,k+1} - a_{n,k}\}|x_k - y_n|^2 \\ &\leqslant \{2a_{n,0} + 2\gamma_n\}\operatorname{diam}(C)^2 \longrightarrow 0. \quad \text{Q.E.D.} \end{aligned}$$

DEFINITION 1. The array $\{a_{n,k}\}$ is said to be proper if for each l^∞*-element* $\{\beta(k)\}$ such that $\Sigma_k a_{n,k} \beta(k) \longrightarrow \delta$, then $\Sigma_{k,l} a_{n,k} a_{n,l} \beta(|k - l|) \longrightarrow \delta$.

Cesàro means are proper in this sense, as we can see from a simple computation, as are other familiar summation methods.

THEOREM 2. *Suppose in Theorem* 1 *that* $0 \in C$, $T(0) = 0$ *and that for some* $c \geqslant 0$, T *satisfies for all* u, v *the inequality*

$$\text{(i)} \qquad |Tu + Tv|^2 \leqslant |u + v|^2 + c\{|u|^2 - |Tu|^2 + |v|^2 - |Tv|^2\}.$$

Suppose that $\{a_{n,k}\}$ *is proper in the sense of Definition* 1 *and that*

$$\sum_{k \geqslant 0} |a_{n,k+1} - a_{n,k}| \longrightarrow 0 \quad (n \longrightarrow +\infty).$$

Then $A_n(x)$ *converges strongly.*

Obviously (i) will hold with $c = 0$ if $C = -C$ and T is odd.

LEMMA 2. *Let $\{x_j\}$ be a bounded infinite sequence in H, $|x_j| \leqslant d_0$, $y_n = \Sigma_{k \geqslant 0} a_{n,k} x_k$ where $\{a_{n,k}\}$ is an array as in the hypothesis of Theorem 2. Suppose that y_k converges weakly to y and that (x_j, x_{j+k}) converges to $q(k)$ as $j \longrightarrow +\infty$, uniformly in k. Then y_n converges strongly to y.*

PROOF OF LEMMA 2. We first show that $\Sigma_{k \geqslant 0} a_{n,k} q(k) \longrightarrow |y|^2$. Let $\epsilon > 0$ be given. We may find $j(\epsilon)$ such that for $j \geqslant j(\epsilon)$ and all k, $|(x_j, x_{j+k}) - q(k)| < \epsilon$. We note that

$$\left|(x_j, y) - \sum_{k \geqslant 0} a_{n,k} q(k)\right| \leqslant |(x_j, y - y_n)| + \sum_{k=0}^{j-1} a_{n,k} d_0^2$$
$$+ \sum_{k \geqslant 0} |a_{n,k+j} - a_{n,k}||(x_j, x_{j+k})| + \sum_{k \geqslant 0} a_{n,k} |q(k) - (x_j, x_{j+k})|.$$

If we choose $j \geqslant j(\epsilon)$ and then $n \geqslant n(\epsilon, j)$, it follows that we can make $|(x_j, y) - \Sigma_{k \geqslant 0} a_{n,k} q(k)| < 2\epsilon$. Hence for $j, j_1 \geqslant j(\epsilon)$, $|(x_j, y) - (x_{j_1}, y)| < 4\epsilon$. Thus (x_j, y) converges as $j \longrightarrow \infty$, and since $\Sigma_{j \geqslant 0} a_{n,j}(x_j, y) = (y_n, y)$ must converge to the same limit, $(x_j, y) \longrightarrow |y|^2$. Thus for a large choice of j and $n \geqslant n(\epsilon, j)$,

$$\left|\sum_{k \geqslant 0} a_{n,k} q(k) - |y|^2\right| < 4\epsilon.$$

i.e., $\Sigma_{k \geqslant 0} a_{n,k} q(k) \longrightarrow |y|^2$.

To prove the lemma, it suffices to show that $\overline{\lim} |y_n|^2 \leqslant |y|^2$. We have $|y_n|^2 = \Sigma_{k,l \geqslant 0} a_{n,k} a_{n,l} (x_k, x_l)$. From the assumption of Lemma 2, we have $|(x_k, x_l) - q(|k - l|)| < \epsilon_{\min(k,l)}$ where $\epsilon_p \longrightarrow 0$ as $p \longrightarrow \infty$. Therefore

$$|y_n|^2 \leqslant \sum_{k,l \geqslant 0} a_{n,k} a_{n,l} q(|k - l|) + \sum_{k,l \geqslant 0} a_{n,k} a_{n,l} \epsilon_{\min(k,l)}$$
$$\leqslant \sum_{k,l \geqslant 0} a_{n,k} a_{n,l} q(|k - l|) + 2 \sum_{k \geqslant 0} a_{n,k} \epsilon_k.$$

Since $a_{n,k}$ is proper we conclude that $\overline{\lim} |y_n|^2 \leqslant |y|^2$. Q.E.D.

PROOF OF THEOREM 2. It follows from the inequality (i) that $|(x_{j+s}, x_{j+k+s}) - (x_j, x_{j+k})| \leqslant (c + 1)\{|x_j|^2 - q(0)\} \longrightarrow 0$ as $j \longrightarrow +\infty$. Hence we may apply Lemma 2 to obtain Theorem 2. Q.E.D.

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