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INDEPENDENT KNOTS IN BIRKHOFF INTERPOLATION[1]

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We consider Birkhoff interpolation for an incidence matrix $E = (e_{ik})_{i=1}^{m}{}_{;}{}_{k=0}^{n}$, the "polynomials" $P = \Sigma_0^n a_k u_k(x)$, for a system $U = \{u_k\}_0^n$ of functions $u_k \in C^n[a, b]$ (or $P = \{x^k\}_0^n$) and the knots $X = (x_1, \ldots, x_m)$ satisfying $a \leqslant x_1 < \cdots < x_m \leqslant b$. The method of independent knots appears for the first time in **[4]**; it is somewhat related to the coalescence method **[1]**, **[3]**.

A function $f \in C^n[a, b]$ is annihilated by E, X if

$$(1) \qquad f^{(k)}(x_i) = 0 \quad \text{for all } (i, k) \text{ with } e_{ik} = 1.$$

From zeros of f and its derivatives given by (1), one can derive further zeros by means of Rolle's theorem. This leads to the following definition. A *Rolle set* $\mathcal{R}$ for a function f annihilated by E, X is a collection $\mathcal{R}_k$, $k = 0, \ldots, n$, of Rolle sets of zeros (with multiplicities) of the $f^{(k)}$. The sets $\mathcal{R}_k$ are defined inductively: $\mathcal{R}_0$ consists of the zeros of f given by (1); if $\mathcal{R}_0, \ldots, \mathcal{R}_k$ have been defined, we select $\mathcal{R}_{k+1}$—some of the zeros of $f^{(k+1)}$—as follows: (α) $\mathcal{R}_{k+1}$ contains all zeros of $f^{(k)}$ of multiplicity > 1, their multiplicities reduced by 1. (β) $\mathcal{R}_{k+1}$ contains all zeros of $f^{(k+1)}$ (with multiplicities) given by (1). (γ) For any two adjacent zeros $\alpha, \beta \in \mathcal{R}_k$ we select a zero γ of $f^{(k+1)}$ by means of Rolle's theorem, *provided one exists not listed in* (1). This new zero γ may be different from the x_i; it may be one of the x_i, but not listed in (1) as a zero of $f^{(k+1)}$; finally, γ may appear as an additional multiplicity of a zero x_i of $f^{(k+1)}$ by (1). In this case, $e_{i,k+1} = \cdots = e_{i,k+t} = 1$, $e_{i,k+t+1} = 0$. If no zero γ as specified exists, there is a *loss*. (δ) We adjust the multiplicities in the last case of (γ): if also $e_{i,k+t+2} = \cdots = e_{i,k+s+1} = 0$, then γ belongs to $\mathcal{R}_{k+1}$ with multiplicity s. A Rolle set constructed without losses is *maximal*. A function f annihilated by E, X may have several Rolle sets, some of them maximal, others are not. Let m_k be the number of ones in the column k of E, let

$$(2) \qquad \mu_k = (\cdots((m_0 - 1)_+ + m_1 - 1)_+ + \cdots + m_{k-1} - 1)_+ + m_k.$$

LEMMA 1. *The number of distinct Rolle zeros of $f^{(k)}$ in a maximal Rolle set is exactly μ_k.*

Let E be a Birkhoff matrix, let E^0 be derived from E by replacing a one,

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$e_{i_0 q} = 1$, $1 < i_0 < m$ by zero, let E', E'' consist of rows $1, \ldots, i_0$ and $i_0, \ldots, m$ of E^0. Let μ_k^0, μ_k', μ_k'' be defined for the matrices by (2).

LEMMA 2. *If $e_{i_0 q} = 1$, one has $\tau = \mu_q^0 - \mu_q' - \mu_q'' \geqslant 0$, and if, in addition, $e_{i_0 - 1, q} = 0$, then $\tau > 0$.*

A set $Y \subset [a, b]$ is *independent* with respect to $\mathcal{U}$ if for each $X \subset Y$, each polynomial P annihilated by E, X has a maximal Rolle set. Results on independent sets are based on inequalities of Markov type and on

LEMMA 3. *For each $l > 0$ there is a number $d > 0$ with the property that if $P(\alpha) = P(\beta) = 0$, $\beta - \alpha > l$, then at least one point $\alpha + d < \xi < \beta - d$ satisfies $P'(\xi) = 0$.*

THEOREM 1. *There exist independent sets $Y = \{y_s\}_{s=-\infty}^{+\infty}$ so that $a < \cdots < y_{-s} < \cdots < y_s < \cdots < b$; the y_s can be defined inductively; at each step it is enough to take y_s (or y_{-s}) sufficiently close to b (or to a).*

THEOREM 2. *If $Y = \{y_s\}_{s=-\infty}^{+\infty}$ is an independent set, there exist points z_{st} in (y_s, y_{s+1}) so that the set formed by all z_{st} and all y_s is independent.*

LEMMA 3. *Let $1 \leqslant s < i_0 < t \leqslant m$. There exists an independent set $(x_1, \ldots, x_m)$ and an interval $I = [c, d] \subset (x_{i_0 - 1}, x_{i_0 + 1})$ so that:* (i) *If P is annihilated by E, X, then Rolle zeros of P are derived only from x_i, $s \leqslant i \leqslant t$;* (ii) *problem* (1) *for E, X is regular if $x_{i_0} \in I$, and row i_0 of E is consevrative.*

By means of these results we can estimate the number of changes of signs of determinants $D_E(X)$ of (1). Let $\mathcal{U} = \mathcal{P}$.

THEOREM 3. *If X is as in Lemma* 3, *and if row i_0 of E has exactly one odd supported sequence beginning with $e_{i_0, q} = 1$, then, as x_{i_0} moves from c to d, $D_E(X)$ changes sign at least τ times. If X', X'' have x_{i_0} in the extreme positions,*

$$\operatorname{sign} D_E(X') = (-1)^\tau \operatorname{sign} D_E(X''). \tag{3}$$

COROLLARIES. 1. *If E is a Birkhoff matrix, $s = 1$, $t = m$, then $\tau > 0$ and E is strongly singular. This is the main theorem of* [2], *but with a precise number of changes of sign.*

2. *Assume that row i_0 consists of disjoint portions S_j, $j = 1, \ldots, p$, which follow each other. Let matrix E_j have rows $s, \ldots, t$ of E, with row i_0 replaced by three rows $S_1 \cup \cdots \cup S_{j-1}$, S_j, and $S_{j+1} \cup \cdots \cup S_p$. Let S_j have exactly one odd supported sequence in E_j with τ_j constructed as in Lemma 2. Then E is strongly singular if $\Sigma\tau_j + \sigma$ is odd, where σ is the difference of the interchanges of rows for the two coalescences $(\cdots(S_1 \cup S_2) \cup \cdots \cup S_p) \cup E_{i_0+1}$ and $S_1 \cup (S_2 \cup \cdots \cup (S_p \cup E_{i_0+1}) \cdots)$. If $s = 1$, $t = m$, this is the criterion* [1, Theorem 2.3].

3. *If* $s \neq 1$, $t \neq m$, *we obtain new criteria. One notices the phenomenon that a submatrix* F *of* E *may be "so bad" that any of its extensions to a Birkoff matrix is strongly singular.*

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