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LOCAL GEVREY AND QUASI-ANALYTIC HYPOELLIPTICITY FOR $\Box_b$

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Introduction. The $\bar{\partial}_b$ complex is well defined on any smooth *CR* manifold *M*, and once a metric is fixed, so is the complex Laplace-Beltrami operator $\Box_b$ on forms of type (p, q). For compact *M* without boundary, the $\bar{\partial}_b$ cohomology of *M* may be studied via $\Box_b$ [6], and thus local smoothness of solutions to $\Box_b u = f$ is important. In its own right, $\Box_b$ is a prototype of doubly characteristic operators. Under suitable convexity conditions on *M*, Kohn [7] established the following subelliptic estimate on (p, q) forms in $C_0^\infty(M)$:

$$\|\varphi\|_{1/2}^2 \leqslant C(\Box_b\varphi, \varphi) + C'\|\varphi\|_0^2,$$

and in general such estimates imply C^∞ and Gevrey (G^s, $s \geqslant 2$) hypoellipticity locally [3], [8], [10] and no more [1]. In the special case of the Heisenberg group, Folland and Stein [5] found an explicit fundamental solution which gives local analytic hypoellipticity; while, in general, if *M* is compact, satisfies the convexity condition $Y(q)$ of Kohn, and has an invertible Levi form, the author proved $\Box_b$ is globally analytic hypoelliptic and so is the $\bar{\partial}$-Neumann problem (joint work with M. Derridj, cf. [4], [9]).

In this note we assume $Y(q)$ and the invertibility of the Levi form and prove local regularity in all Gevrey classes G^s with $s > 1$ as well as in a quasi-analytic class. Full details will appear elsewhere.

Notations and definitions. The class $C^L(\Omega) \subset C^\infty(\Omega)$, Ω open in R^n, is defined by the condition that for all $K \subset\subset \Omega$ there exists a constant $C_{f,K}$ such that for any multi-index α,

$$\sup_K |D^\alpha f| \leqslant C_{f,K}^{|\alpha|+1} L(|\alpha|)^{|\alpha|},$$

where we assume that the sequence $\{L(j)\}$ of positive numbers satisfies (1) $L(j)/j$ is nondecreasing and (2) $L(j+1)^{j+1} \leqslant C^j L(j)^j$ uniformly in j. The second condition implies that $C^L(\Omega)$ is closed under differentiation while the first implies that the class is preserved under composition. Thus one may speak of C^L manifolds. If, in addition, $\Sigma L(j)^{-1} < \infty$, the class is called non-quasi-analytic (NQA) and admits compactly supported functions. Common examples are the Gevrey classes $G^s(\Omega)$, obtained by taking $L(j) = j^s$. These are NQA if $s > 1$, while

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taking $s = 1$ gives the real analytic class. The class obtained by taking $L(j) = j \log j$ is quasi-analytic, and thus contains no compactly supported functions.

A smooth manifold M of dimension $2n - 1$ is called *CR* provided (a) $\mathbf{C}T(M)_x = S_x \oplus \overline{S_x} \oplus F_x$ for all x, S a smooth subbundle of $\mathbf{C}T(M)$ of complex dimension $n - 1$, orthogonal to $\bar{S}$ under a smooth Hermitian inner product which induces a Riemannian metric on M, and F of complex dimension 1, and (b) if Y_1, Y_2 are local sections of S, so is their commutator $[Y_1, Y_2]$. $D^{p,q}(M)$, the space of smooth (p, q) forms on M, is defined to be the space of those smooth $p + q$ forms on M such that $h(t_1, \ldots, t_{p+q}) = 0$ if p of the t's and q of the $\bar{t}$'s have zero projection on S. For $v \in D^{p,q}(M)$ we define $\bar{\partial}_b v$ to be the projection on $D^{p,q+1}(M)$ of dv. The $\bar{\partial}_b$ form a complex and we denote by $\Box_b$: $D^{p,q}(M) \longrightarrow D^{p,q}(M)$ the operator $\bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$, where $\bar{\partial}_b^*$ denotes the formal L^2 adjoint of $\bar{\partial}_b$. When $Y_1, \ldots, Y_{n-1}$ forms a local frame for S and T denotes a local, nowhere zero, purely imaginary section of F, the matrix c_{ij}, given by $[Y_i, \bar{Y}_j] \equiv c_{ij} T$ modulo $S \oplus \bar{S}$, is the Levi form of M. The number of its nonzero eigenvalues and its signature in absolute value are independent of the choice of Y_j and of T. M satisfies $Y(q)$ if c_{ij} has $\max(q + 1, n - q)$ eigenvalues of the same sign or pairs of eigenvalues of opposite signs. M is strictly pseudoconvex if all eigenvalues are strictly of the same sign.

Results.

THEOREM. *Let M be a CR manifold of class C^L, L satisfying* (1) *and* (2) *above and non-quasi-analytic, of dimension $2n - 1$ with an invertible Levi form satisfying $Y(q)$ in an open set Ω. Then any $u \in D^{p,q}(\Omega)$ satisfying $\Box_b u \in C^L(\Omega)$ is itself in $C^L(\Omega)$.*

PROPOSITION (BOMAN, CF. [2]). *The intersection of all $C^L(\Omega)$, L satisfying* (1) *and* (2) *and non-quasi-analytic, is the class $C^{L'}$ where $L'(j) = j \log j$.*

COROLLARY. *Let M be a CR manifold of the quasi analytic class $C^{L'}$, $L'(j) = j \log j$, with invertible Levi form satisfying $Y(q)$. Then any $u \in D^{p,q}(\Omega)$, Ω open in M, with $\Box_b u \in C^{L'}(\Omega)$, is itself in this class.*

Remarks. It is well known that one need only assume that $u \in D'(\Omega)$ for the above results to hold [6], [8]. Also, there is a direct proof of the Corollary which obviates the quasi-analyticity of $C^{L'}$ and obtains the local result by considering a family of compactly supported functions whose derivatives, up to a given order, grow uniformly as if the functions belonged to $C^{L'}$. Families of this sort were introduced by Ehrenpreis to localize some real analytic problems; while his families fail to satisfy an analogue of (1) above, one may approximate $C^{L'}$ by families which do.

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