SIMPLE RIEMANN FUNCTIONS

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Research of Cohn [1] and Daggit [2] guesses or deduces functions M and G such that some Riemann functions have the form $R = Mf(G)$. The Riemann function is then found by solving an ordinary differential equation for f. Roughly speaking, all the results of Riemann, Cohn and Daggit are *simple* Riemann functions because (i) the equation is almost selfadjoint, (ii) the ordinary differential equation for f has coefficients depending on G, and (iii) the characteristic conditions for the Riemann function become initial conditions for f. These restrictions, formulated in Definition 2, seem mild because omitting any one of the three would allow every Riemann function to satisfy Definition 2.

Surprisingly, these conditions lead directly to expressions for M and G (Theorems 1 and 2). With these, *all* of Daggit's results are derived (Theorems 3 to 7) from *one* of his hypotheses. For a *selfadjoint* equation to have a simple Riemann function, Daggit's hypothesis is sufficient as well as necessary (Theorem 8). This announcement therefore contains proof that all *simple Riemann functions are known for selfadjoint equations*. The proofs for the general case are given in another paper [4].

DEFINITION 1. The unique *Riemann function R* for the equation

$$(1.1) \qquad U_{xy} + A(x, y)U_x + B(x, y)U_y + C(x, y)U = 0$$

satisfies the equation adjoint to (1.1), depends on two parameters X and Y, reduces to 1 at the point $(x, y) = (X, Y)$, and obeys

$$(1.2) \qquad R_x = BR \text{ when } y = Y \quad \text{and} \quad R_y = AR \text{ when } x = X.$$

When a *multiplier M* is chosen, substituting $R = M(x, y, X, Y) \times U(x, y, X, Y)$ into the equation adjoint to (1.1) gives

$$(1.3) \quad U_{xy} + a(x, y, X, Y)U_x + b(x, y, X, Y)U_y + c(x, y, X, Y)U = 0.$$

If a *guess G* is then chosen, substituting $U = f(G(x, y, X, Y), X, Y)$ into (1.3) yields

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(1.4) $\alpha(x, y, X, Y)f'' + \beta(x, y, X, Y)f' + f = 0.$

DEFINITION 2. The Riemann function for (1.1) is a *simple Riemann function* if it has the form $R = M(x, y, X, Y)f(G(x, y, X, Y), X, Y)$ and for this M and G, (i) the coefficients in (1.3) satisfy $a_x = -ab$, (ii) the coefficient α of (1.4) satisfies $\partial(\alpha, G)/\partial(x, y) = 0$, which implies $\partial(\beta, G)/\partial(x, y) = 0$, and (iii) $G(x, Y, X, Y) = G(X, y, X, Y)$, which is finite (otherwise G could be replaced by $1/G$).

THEOREM 1. *Any M is a function of X and Y times*

(1.5) $M = \exp\left(\int_Y^y A(x, t)\, dt + \int_X^x B(s, Y)\, ds\right),$

and determines $a = 0$, $b = \int_Y^y A_x(x, t) - B_y(x, t)\, dt$, *and* $c = C - AB - B_y$.

PROOF FOR $A = B = 0$. M is a fixed function of X and Y when $x = X$ or $y = Y$, because both R and G have this property. M also satisfies the equation $M_{xy} = 0$ that results from imposing the condition $a_x = -ab$ on the coefficients $a = M_y/M$, $b = M_x/M$ and $c = C + M_{xy}/M$. The formula given for M is the solution of this Goursat problem.

REMARK. Daggit introduces this multiplier [2, p. 93].

LEMMA 1. *No generality is lost by assuming* $G_{xy} + bG_y = 0$.

PROOF FOR $A = B = 0$. Since b is zero and $G_x G_y/G_{xy} = \alpha/\beta$ is a function of G by condition (ii), the lemma is just a restatement, in complex characteristic coordinates, of the classical result "A function F has the same level curves as some harmonic function if $\nabla^2 F/(\nabla F)^2$ is a function of F" [3, p. 228].

REMARK. Cohn also proves Lemma 1 for $A = B = 0$.

THEOREM 2. *Any G must be functionally dependent on*

(1.6) $G = \int_Y^y \int_X^x c(s, t)\exp\left(\int_X^s b(\theta, t, Y)\, d\theta\right) ds\, dt.$

PROOF FOR $A = B = 0$. Since G_{xy} is zero and $\alpha = (G_x G_y)/c$ is a function of G, say, $1/g''(G)$, it follows that $g(G)$ satisfies $[g(G)]_{xy} = c$. But the only solution of this equation satisfying condition (iii) is $g(G) = \int_Y^y \int_X^x c(s, t)\, ds\, dt$ (plus an inconsequential term depending only X and Y).

REMARK. Equation (1.6) with $b = 0$ is the basic assumption of Cohn's research on selfadjoint equations.

Theorems 3 to 7 are based on *one* of the conditions necessary for Daggit's method [4, Theorem 2, p. 96, Lemma A, p. 98]: namely, that c, $(\ln c)_{xy}$, and b_y are proportional. This hypothesis is invariant under all changes of variables that preserve the canonical form of (1.1).

THEOREM 3. *If $c = 0$ or $c - b_y = 0$, then* (1.1) *has a simple Riemann function given by* $R = M(x, y, X, Y)$ *or* $R = 1/M(X, Y, x, y)$, *respectively*.

THEOREM 4. *If* $b_y = 0$, $c \neq 0$, *and* $(\ln c)_{xy} = 0$, *then* (1.1) *has a simple Riemann function*,

$$R = MJ_0(2\sqrt{U}), \quad \text{where } U = \int_Y^y \int_X^x c(s, t) \, ds \, dt.$$

THEOREM 5. *If* $(\ln c)_{xy} = 0$ *and* b_y/c *is a nonzero constant* $-1/k$, *then* (1.1) *has a simple Riemann function, with* $R/M = L_k(U)$, *a Laguerre function of* $U = -\ln[M(x, y, X, Y)M(X, Y, x, y)]$.

THEOREM 6. *Assume* $c \neq 0$, *and* $b_y = 0$. *Denote*

$$[c(x, y)c(x_0, y_0)] / [c(x_0, y_0)c(x, y)]$$

by Q. *If there exists a constant* k_2 *such that* $1 = \frac{1}{2}k_2(\ln c)_{xy}/c$, *then* (1.1) *has a simple Riemann function*,

$$R = M \, {}_2F_1(p, q; 1; 1 - Q^{-1/2}),$$

where p *and* q *are the roots of* $x^2 - x - k_2 = 0$.

THEOREM 7. *Let* $b_y \neq 0$. *Assume* $c \neq 0$ *satisfies*

$$1 = k_1 \ln(Q)/\ln[M(x, y, X, Y)M(X, Y, x, y)]$$

for some constant k_1. *If* b_y/c *is a constant* k_1/k_2, *then* (1.1) *has a simple Riemann function*, $R = M \, {}_2F_1(p, q; 1; 1 - Q^{-1/2})$, *where* p *and* q *are the roots of* $x^2 + (k_1 - 1)x - k_2 = 0$.

THEOREM 8. *When* $b_y = 0$, (1.1) *has a simple Riemann function if and only if* $(\ln c)_{xy}$ *and* c *are proportional*.

PROOF FOR $A = B = 0$. If (1.1) has a simple Riemann function, it is a function of $G = \int_Y^y \int_X^x c(s, t) \, ds \, dt$, by Theorem 2, and $\partial(G_x G_y/c, G)/\partial(x, y)$ is zero by Definition 1. Using these conditions, Cohn has shown [1] that $(\ln c)_{xy}$ and c are proportional. Conversely, Cohn actually exhibits [1] the required simple Riemann functions when $(\ln c)_{xy}$ and c are proportional.

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