BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 5, September 1976

ON THE TOPOLOGICAL STRUCTURE OF SIMPLY-CONNECTED ALGEBRAIC SURFACES

BY RICHARD MANDELBAUM AND BORIS MOISHEZON

Communicated by S. S. Chern, April 8, 1976

Suppose X is a smooth *simply-connected* compact 4-manifold. Let $P = CP^2$ and $Q = -CP^2$ be the complex projective plane with orientation opposite to the usual. We shall say that X is completely decomposable if there exist integers a, b such that X is diffeomorphic to $aP \# bQ$.

By a result of Wall **[W1]** there always exists an integer k such that $X \# (k+1)P \# kQ$ is completely decomposable. If $X \# P$ is completely decomposable we shall say that X is almost completely decomposable. In **[MM1]** we demonstrated that any nonsingular hypersurface of CP^3 is almost completely decomposable. In this paper we first announce generalizations of this result in two directions as follows.

THEOREM 1. *Suppose W is a simply-connected nonsingular complex projective 3-fold. Then there exists an integer $m_0 \geqslant 1$ such that any hypersurface section V_m of W of degree $m \geqslant m_0$ which is nonsingular will be almost completely decomposable.*

THEOREM 2. *Let V be a nonsingular complex algebraic surface which is a complete intersection. Then V is almost completely decomposable.*

IDEA OF PROOF. The idea of the proofs is to degenerate V (or V_m) to a pair of "less complicated" nonsingular surfaces crossing transversely and then use induction. The topological analysis of such a situation is then taken care of by Corollary 2.5 of **[MM2]** which states:

COROLLARY. *Suppose W is a compact complex manifold and V, X_1, X_2 are closed complex submanifolds with normal crossing in W. Let $S = X_1 \cap X_2$ and $C = V \cap S$ and suppose as divisors V is linearly equivalent to $X_1 + X_2$. Let $\sigma: X_2' \longrightarrow X_2$ be the monoidal transformation of X_2 with center C. Let S' be the strict image of S in X_2' and let $T_2' \longrightarrow S'$, $T_1 \longrightarrow S$ be tubular neighborhoods of S' in X_2' and S in X_1, respectively, with $H_1 = \partial T_1$* and *$H_2' = \partial T_2'$.*

Then there exists a bundle isomorphism $\eta: H_2' \longrightarrow H_2$ which reverses orientation on fibers such that V is diffeomorphic to $\overline{X_2' - T_2'} \cup_\eta \overline{X_1 - T_1}$.

Then if V, X_1, X_2 are simply connected 4-manifolds we can use **[M]** to

---

AMS (MOS) subject classifications (1970). Primary 57D55, 57A15, 14J99.

conclude that $V \approx X_1 \# X_2 \# (n-1)Q \# 2g(P \# Q)$ where $n = \text{card } C$ and $g =$ genus S. The proofs of Theorems 1 and 2 then conclude using an inductive argument.

This method gives us other results for which we establish some additional terminology. A field F is called an algebraic function field of two variables over $\mathbf{C}$ if F is a finitely-generated extension of $\mathbf{C}$ of transcendence degree two. Let $\mathfrak{F}$ denote the collection of all such fields. Then for $F \in \mathfrak{F}$ there exists a nonsingular algebraic surface whose field of meromorphic functions is F (see [Z]). We shall call any such nonsingular surface a model for F. It is then easy to see that given any two such models V_1, V_2 for F their fundamental groups are isomorphic. Thus we define the fundamental group $\pi_1(F)$ for any $F \in \mathfrak{F}$ as the fundamental group of any model V for F. We then let $\mathfrak{F}_0$ be the subcollection of simply-connected F in $\mathfrak{F}$. For $F \in \mathfrak{F}_0$ we let $\mu(F) = \inf\{k \mid \exists$ a model V for F such that $V \# kP$ is completely decomposable$\}$. Using Wall's result previously mentioned, it can be seen that $\mu(F)$ is finite for any $F \in \mathfrak{F}_0$. If F is a pure transcendental extension of $\mathbf{C}$, $\mu(F) = 0$. If $\mu(F) \leqslant 1$ we shall call F a topologically normal field. We now need

DEFINITION 3. Let $L, K \in \mathfrak{F}$. Then L is a satisfactory cyclic extension of K if there exist models V_L, V_K for L, resp. K and a morphism $\Phi\colon V_L \longrightarrow V_K$ with discrete fibers whose ramification locus R_Φ is a nonsingular hypersurface section of V_K whose degree is a multiple of $\deg(\Phi)$.

We then state

THEOREM 4. *Let $K \in \mathfrak{F}_0$. Then there exists a satisfactory cyclic extension $L \in \mathfrak{F}_0$ of K which is of degree 2 over K and topologically normal.*

In [M] it is further shown that if K itself is topologically normal then so is any satisfactory cyclic extension. These two results motivate a partial order in $\mathfrak{F}_0$ defined as follows:

For $L, K \in \mathfrak{F}_0$ we shall say that L is a satisfactorily resolvable extension of K iff there exists a finite sequence of fields $L_0, \ldots, L_n$ in $\mathfrak{F}_0$ with $L_0 = K$, L_{i+1} a satisfactory cyclic extension of L_i and $L_n = L$. We write $K < L$ if L is a satisfactorily resolvable extension of K. Then $<$ induces a partial ordering on $\mathfrak{F}_0$. Our above results then say that in terms of this partial ordering every sufficiently "large" field L is topologically normal.

Lastly we mention a purely topological counterpart of Theorem 4.

THEOREM 5. *Suppose X is a smooth simply-connected 4-manifold. Let $\mathfrak{F} \in H_2(X, \mathbf{Z})$ with $\mathfrak{F}^2 \neq 0$ and $\mathfrak{F}$ divisible by some integer $m \geqslant 2$. Then there exists a smooth compact simply connected 4-manifold $\widetilde{X}$ and a map $\Phi\colon \widetilde{X} \longrightarrow X$ exhibiting $\widetilde{X}$ as an m-fold branched cover over X whose branch locus R is a nonsingular representative of $\mathfrak{F}$ such that*

(1) *If* $F^2 > 0$ *then* $\widetilde{X} \# P$ *is completely decomposable.*

(2) *If* $F^2 < 0$ *then* $\widetilde{X} \# Q$ *is completely decomposable.*

BIBLIOGRAPHY

[M] R. Mandelbaum, *Irrational connected sums and the topology of algebraic surfaces* (to appear).

[MM1] R. Mandelbaum and B. Moishezon, *On the topological structure of non-singular algebraic surfaces in* CP^3, Topology **15** (1976), 23–40.

[MM2] ———, *On the topology of simply-connected algebraic surfaces* (to appear).

[W1] C. T. C. Wall, *Diffeomorphisms of 4-manifolds*, J. London Math. Soc. **39** (1964), 131–140. MR **29** #626.

[W2] ———, *On simply-connected 4-manifolds*, J. London Math. Soc. **39** (1964), 141–149. MR **29** #627.

[Z] O. Zariski, *Algebraic surfaces*, Ergebnisse Math. Grenzgebiete, Band 61, 2nd ed., Springer-Verlag, Berlin and New York, 1971.

DEPARTMENT OF MATHEMATICS, WEIZMANN INSTITUTE, REHOVOT, ISRAEL

DEPARTMENT OF MATHEMATICS, TEL AVIV UNIVERSITY, RAMAT-AVIV, ISRAEL