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ON SURFACES OBTAINED FROM QUATERNION ALGEBRAS OVER REAL QUADRATIC FIELDS[1]

BY IRA SHAVEL

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Let A be a totally indefinite division quaternion algebra with center $k = \mathbf{Q}(\sqrt{d})$, $d > 0$, $\mathcal{O}$ a maximal order in A, and $\Gamma(1) = \{\alpha \in \mathcal{O} \mid \nu(\alpha) = 1\}$ where ν is the reduced norm from A to k. Fix an isomorphism λ such that $A \otimes_Q \mathbf{R} \cong M_2(\mathbf{R}) \oplus M_2(\mathbf{R})$. Then $\lambda(\Gamma(1) \otimes_Q 1) \subseteq \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$, and $j(\Gamma(1)) = \Gamma(1)/(\text{center } \Gamma(1))$ acts holomorphically and properly discontinuously on $X = H \times H$, where H is the usual upper half plane. In general, if Γ is any group of holomorphic automorphisms of X acting properly discontinuously and without fixed points, then $\Gamma\backslash X$ is a complex manifold. Since A is division the quotient is compact, and it is known to be a projective algebraic variety. In this note we discuss the numerical invariants and second cohomology group of $U(\Gamma) = \Gamma\backslash H \times H$ where Γ is commensurable with $\Gamma(1)$.

(A) For any algebraic number field F, a quaternion algebra with center F is determined up to isomorphism by a finite set $S(A)$ of prime divisors of F. Denote this algebra by $A(F, S(A))$.

THEOREM 1. *Assume $h(k)$ = class number of $k = 1$. Let $j(\Gamma(1)) = \Gamma(1)/\{\pm 1\}$, $A = A(k, S(A))$, and let*

$$\left(\frac{-}{p}\right)$$

be the Kronecker symbol. $j(\Gamma(1))$ acts on X without fixed points $\Leftrightarrow$ all of the following hold:

$$\left(\frac{-3}{p}\right) = 1 \quad or \quad \left(\frac{-D}{p}\right) = 1 \tag{1}$$

for some $P \in S(A)$, where $p\mathbf{Z} = P \cap \mathbf{Z}$ and $-D'$ is the discriminant of the field $\mathbf{Q}(\sqrt{-3d})$.

$$\left(\frac{-1}{p}\right) = 1 \ or \ \left(\frac{-D'}{p}\right) = 1 \tag{2}$$

for some $P \in S(A)$, where $p\mathbf{Z} = P \cap \mathbf{Z}$ and $-D'$ is the discriminant of the field $\mathbf{Q}(\sqrt{-d})$.

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[1] Partial results of the author's dissertation [3] under M. Kuga.

(3) *If* $d = 5$, $\exists P \in S(A)$ *such that* $p\mathbf{Z} = P \cap \mathbf{Z}$ *and* $p \equiv 1 \pmod 5$.

Let $A^{X++} = \{\alpha \in A^X \mid \nu(\alpha)$ is totally positive$\}$ and call such α totally positive. Let $E^{++} = \mathcal{O}^X \cap A^{X++}$. $|j(E^{++}): j(\Gamma(1))| = 2$ if ϵ_k, the fundamental unit of k greater than 1, is totally positive, and $|j(E^{++}): j(\Gamma(1))| = 1$ otherwise.

THEOREM 2. *Assume* $h(k) = 1$ *and* ϵ_k *is totally positive.* $j(E^{++})$ *acts on* X *without fixed points* $\Leftrightarrow$ *both of the following hold*:

(1) $j(\Gamma(1))$ *has no elements of finite order.*

(2) $\exists P \in S(A)$ *such that* P *splits in* $k(\sqrt{-\epsilon_k})/k$.

Consider $B^{++} = \{\beta \in A^{X++} \mid \beta\mathcal{O} = \mathcal{O}\beta\}$ = normalizer of $\Gamma(1)$ in A^{X++}. If $h(k) = 1$ then the class number of a maximal order in A is also 1. Therefore every 2-sided $\mathcal{O}$-ideal is principal. The set of all 2-sided maximal $\mathcal{O}$-ideals are in one-to-one correspondence with the prime ideals of $\mathcal{O}_k$. Let $\mathcal{P}_i = \Pi_i\mathcal{O}$ correspond to $P_i = \pi_i\mathcal{O}_k$.

THEOREM 3. *Assume* $h(k) = 1$. *Let* ϵ *be a fundamental unit of* $\mathcal{O}_k$. *Let* $\{\pi_i\}_{i=1,2,\dots,n}$ *correspond to* $\{\Pi_i\mathcal{O}\}_{i=1,2,\dots,n} = S(A)$. *For these* π_i *let* $\eta(i_1, i_2, \dots, i_r) = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_r}$ *where* $\pi_{i_s} \neq \pi_{i_t}$ *for* $s \neq t$. $j(B^{++})$ *acts on* X *without fixed points if and only if both of the following hold*:

(1) $j(E^{++})$ *has no elements of finite order.*

(2) *For all totally positive* $\eta(i_1, i_2, \dots, i_r)$, $\exists P \in S(A)$ *such that* P *splits in* $k(\sqrt{-\eta(i_1, i_2, \dots, i_r)})/k$, *and for all totally positive* $\eta(i_1, i_2, \dots, i_r)\epsilon$ *(for some choice of* ϵ*)*, $\exists P \in S(A)$ *such that* P *splits in* $k(\sqrt{-\eta(i_1, i_2, \dots, i_r)\epsilon})/k$.

(B) Throughout this section Γ is a group commensurable with $j(\Gamma(1))$ acting on X without fixed points. Using a result of Matsushima and Shimura [2] we have

PROPOSITION 1. (1) *The Euler characteristic* E, *the geometric genus* p_g, *and the arithmetic genus* p_a *of* $\Gamma\backslash X$ *have the following relationship*: $E = 4(p_g + 1) = 4p_a$.

(2) *The irregularity* q *is* 0.

(3) *Then mth plurigenus* $P_m = (p_g + 1)(2m - 1)^2$, $m \geqslant 2$.

COROLLARY. $\Gamma\backslash X$ *is a surface of general type.*

Using the Riemann-Roch theorem we have

COROLLARY. $c_1^2 = 8p_g + 8$, *where* c_1 *is the first Chern class of* $\Gamma\backslash X$.

Using a formula of Shimizu [4] for the volume of a fundamental domain for the action of $j(\Gamma(1))$ on X, and the Gauss-Bonnet theorem we obtain

THEOREM 4. $E(U(1))$, *the Euler characteristic of* $j(\Gamma(1))\backslash X$ *is given by*

$$E(U(1)) = \frac{B_d}{12} \prod_{P \in S(A)} (N_{k/Q}P - 1)$$

where B_d is the generalized Bernoulli number of the numerical character modulo d associated to the field $k = \mathbf{Q}(\sqrt{d})$.

For $d \neq 5$, B_d is an integer. With the aid of a computer, James Maiorana has calculated B_d *for* $d < 750$.

We have a complete list of surfaces with $p_g = 0$ and $p_g = 1$ which come from groups Γ, $j(\Gamma(1)) \subseteq \Gamma \subseteq j(B^{++})$.

(c) Let $U(1) = j(\Gamma(1))\backslash X$ be an algebraic variety. $H_1(U(1), \mathbf{Z})$ is isomorphic to $H^2(U(1), \mathbf{Z})_{\text{torsion}}$ by Poincaré and Pontrjagin duality. Thus

$$H^2(U(1), \mathbf{Z})_{\text{tor}} \cong j(\Gamma(1))/[j(\Gamma(1)), j(\Gamma(1))] \cong \Gamma(1)/\{\pm 1\}[\Gamma(1), \Gamma(1)].$$

By constructing a normal subgroup of $\Gamma(1)$ containing $[\Gamma(1), \Gamma(1)]$, we obtain

THEOREM 5. *Let $j(\Gamma(1))$ act on X without fixed points. Then $|H^2(U(1), \mathbf{Z})_{\text{tor}}|$ is divisible by $a \cdot b \cdot c \cdot \prod_{P \in S(A)} (N_{k/Q}P + 1)$ where*

$$a = \begin{cases} 1/2 & \text{if } N_{k/Q}P \equiv 1 \pmod 4 \text{ for some } P \in S(A), \\ 1 & \text{otherwise}; \end{cases}$$

$$b = \begin{cases} 4 & \text{if } \exists\, P, Q \text{ such that } P \neq Q,\ PQ = 2\mathbf{Z} \text{ and } P, Q \notin S(A), \\ 2 & \text{if } \exists\, P, Q \text{ such that } PQ = 2\mathbf{Z} \text{ and } P \notin S(A) \text{ but } Q \in S(A), \text{ or if } \exists\, P \text{ such that } P^2 = 2\mathbf{Z} \text{ and } P \notin S(A), \\ 1 & \text{otherwise}; \end{cases}$$

$$c = \begin{cases} 9 & \text{if } \exists\, P, Q \text{ such that } P \neq Q,\ PQ = 3\mathbf{Z} \text{ and } P, Q \notin S(A), \\ 3 & \text{if } \exists\, P, Q \text{ such that } PQ = 3\mathbf{Z} \text{ and } P \notin S(A) \text{ but } Q \in S(A), \text{ or if } \exists\, P \text{ such that } P^2 = 3\mathbf{Z} \text{ and } P \notin S(A), \\ 1 & \text{otherwise.} \end{cases}$$

EXAMPLE. Let $A = A(\mathbf{Q}(\sqrt{5}, \{P_5, P_{31}\}))$. We have $P_5^2 = 5\mathbf{Z}$, $N_{k/Q}P_5 = 5$, $P_{31}P'_{31} = 31\mathbf{Z}$, $N_{k/Q}P_{31} = 31$, $N_{k/Q}P_2 = 4$, $N_{k/Q}P_3 = 9$, $\epsilon_k = (1 + \sqrt{5})/2$, $N_{k/Q}\epsilon_k = -1$, and $B_5 = 4/5$. $U(1) = j(\Gamma(1))\backslash X$ is smooth, $E(U(1)) = (1/12) \cdot (4/5)(5 - 1)(31 - 1) = 8$, so $p_g = 1$. $|H^2(U(1), Z)_{\text{tor}}|$ is divisible by $(1/2)(5 + 1)(31 + 1) = 96$. There are two subgroups between $j(\Gamma(1))$ and $j(B^{++})$ yielding $p_g = 0$ surfaces. For more examples see [3].

(D) Let K be the canonical line bundle of a surface of the above type. In conjunction with Gordon Jenkins, we have shown that in the case $P_g = 0$, $3K$ is very ample, that is, $3K$ determines a biholomorphic imbedding into some complex projective space.

Gordon Jenkins [1] has investigated cases where $[k : \mathbf{Q}] \geqslant 3$.

REFERENCES

1. G. Jenkins, Thesis, SUNY at Stony Brook, 1976.

2. Y. Matsushima and G. Shimura, *On the cohomology groups attached to certain vector valued differential forms on the product of the upper half planes*, Ann. of Math. (2) **78** (1963), 417–449. MR **27** #5274.

3. I. Shavel, Thesis, SUNY at Stony Brook, 1976.

4. H. Shimizu, *On discontinuous groups operating on the product of the upper half planes*, Ann. of Math. (2) **77** (1963), 33–71. MR **26** #2641.

For correction see: H. Shimizu, *On zeta functions of quaternion algebras*, Ann. of Math. (2) **81** (1965), 166–193. MR **30** #1998.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, CENTER AT STONY BROOK, STONY BROOK, NEW YORK 11794