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Markov chains, by D. Revuz, North-Holland Mathematical Library, vol. 11, North-Holland, Amsterdam; American Elsevier, New York, 1975, x + 336 pp., $35.50.

The theory of Markov processes can be considered in a great variety of settings. In the work under review the word "chain" is used to indicate discrete time (a convincing argument for this usage is made), the state space is a general measurable space, and the transition probabilities are assumed to be stationary. This context then determines the problems to be considered.

The most immediate problem, and historically the first to be pursued, concerns the asymptotic behavior of the n-step transition probabilities $P^n(x, A)$. In case the state space consists of a finite number of states only, this reduces to studying the asymptotic behavior of the nth power of a Markov matrix, and much early work was devoted to this situation. The case of general state space is of course much more complicated, and the pioneering work here is due to Doeblin. Between these two levels of generality lies that of denumerable state space, definitively treated by Kolmogorov, and alternatively by Feller.

In the ergodic theory of Markov chains one generally distinguishes between the recurrent and the transient case. Roughly, in the recurrent situation, a subset A of the state space will be visited infinitely often by the Markov chain started at x, for all (or most) starting points x, provided only that A is not too small (in a suitable sense). The parenthetical expressions can be made precise in various ways, leading to very different concepts of recurrence. In the denumerable case one may take "most" to mean all, and "small" to mean void. Following one of Doeblin's approaches for general state space, one can take "small" to mean of φ-measure zero, where φ is an auxiliary measure on the state space. Then taking "most" to mean all, one obtains the notion of φ-recurrence. A chain that is φ-recurrent for some φ is recurrent in the sense of Harris.

A subset A of the state space is closed if $P(x, A) = 1$ for all $x \in A$. No matter what notion of recurrence is used, the first problem is to show that the state space can be broken up into minimal closed sets, and the Markov chain restricted to any one of these sets is either recurrent or transient.

In the recurrent case one hopes to establish that $P^n(x, \cdot)$ is asymptotically independent of x (weak ergodicity); one can then expect convergence of

$P^n(x, \cdot)$ to a limiting measure π (strong ergodicity) if and only if there exists a probability measure π satisfying $\pi = \pi P$ (invariant probability measure). One may consider actual convergence as $n \to \infty$, or Cesàro convergence, and various notions of convergence are of interest. Suppose, in particular, that $P^n(x, \cdot) - P^n(y, \cdot) \to 0$ as $n \to \infty$, for all x and y, the convergence being in the sense of total variation. This very strong kind of weak ergodicity is equivalent to each of the following two conditions: (i) the Markov chain has a trivial tail σ-field, (ii) the only bounded solutions to $h_n(x) = Ph_{n+1}(x)$, $n = 0, 1, \ldots$, are $h_n(x) \equiv$ constant.

Ergodic problems for Markov chains have given rise to a great variety of approaches. Some of the most powerful ones are still inventions by Doeblin. Assuming the recurrence condition of Harris, it turns out, one obtains essentially all the results that hold in the denumerable recurrent case. The principal result asserts that one is, in fact, in the tail trivial situation, or reduces to this situation after accounting for a simple periodicity. A full development is given in Chapter VI. At the other extreme, under very weak recurrence conditions one is in the context of general ergodic theory. The basic result here is the Chacon-Ornstein theorem. This theorem has given rise to a number of different proofs, two of which are presented in Chapter 4. There are, however, many intermediate situations, where more can be said than what is provided by the Chacon-Ornstein theorem, but the recurrence condition of Harris fails. There have been numerous investigations of such problems, but these are not discussed in the book.

It is well known that the differential generator of Brownian motion is the Laplacian, and so the study of Brownian motion is one aspect of potential theory. Similarly, each Markov process has an associated potential theory. In the Markov chain case $(P - I)$ takes the place of the Laplacian. In the transient case $G = \sum_{n=0}^{\infty} P^n$ is the potential kernel. The ideas of potential theory thus become available (Chapter 2). In the transient case the Markov chain must escape to infinity: the study of how it does so can be accomplished by suitably metrizing and completing the state space, obtaining in this way the Martin boundary. This is carried through in Chapter 7 for general state space, under some supplementary hypotheses. In the recurrent case the series defining G diverges. For many purposes one can work instead with the resolvent kernel $G_\lambda = \sum \lambda^n P^n$. However it sometimes is desirable to have an actual potential operator, that is, an operator which in some sense inverts $(I - P)$. The classical model here is the logarithmic potential associated with two dimensional Brownian motion. This topic is discussed in Chapter 8.

A Markov chain with the real line for state space is a random walk if the transition probabilities are translation invariant, so that the transition probability operator is simply convolution with a given probability measure μ. Random walks on more general groups are defined analogously. The study of random walks on Euclidean n-space, or the Euclidean lattice is classical. Most of these results can be extended to locally compact abelian groups with countable basis, whilst partial results exist in the nonabelian case. An exposition is given in Chapter 5. Potential theory for recurrent random walks on abelian groups is developed in Chapter 9. However the assumption that the random walks are recurrent in the sense of Harris rules out, for example, many random walks on the line which are merely interval recurrent. In the exposition of random walks on groups, the author draws in part on his own important contributions. In the transient case a principal result is the renewal theorem. In the case of the real line, for example, this asserts that if the support of μ generates the whole line, then $G(x, \cdot)$ converges to a constant c_+ (c_-) times Lebesgue measure as $x \to \infty$ ($x \to -\infty$) and the constants are identified in terms of μ (e.g. if μ has a mean $\lambda > 0$ then $c_+ = \lambda^{-1}$, $c_- = 0$). At the heart of the renewal theorem is again the absence of nonconstant bounded invariant functions, i.e. if the support of μ generates the line then all bounded continuous solutions of $Ph = h$ are constant.

Two questions should now be asked. First, is the theory now essentially complete? Second, has the introduction of the language of potential theory really advanced the subject?

Indications are that the answer to the first question is in the negative: the subject still seems to be opening up in new and important directions. To mention just two striking examples of very recent progress: the work of Rost, Baxter and Chacon, and others, uncovering the connections between Skorokhod stopping, potential theory, and filling schemes; the novel type of occupation time limit theorems of Donsker and Varadhan.

As to the use of the language of potential theory, it can be admitted that in some instances the contribution is largely semantic. However there are numerous situations where the ideas of potential theory have contributed incisively to understanding the behavior of Markov chains. This is certainly the case in the analysis of the asymptotic behavior of transient Markov chains by means of Martin boundaries, in the solution to the Skorokhod stopping problem just mentioned, and for many other questions as well.

The text of Revuz is supplemented by short historical notes. In this connection it should be noted that the ergodic theorem for chains recurrent in the sense of Harris, which in the text appears credited to the reviewer, was first proved in a joint paper with B. Jamison, as correctly stated in the notes. In view of the familiar process of historical compression, it may not be surprising that Markov's name does not appear in the references; more remarkable is the fact that Doeblin also is not cited. As the author remarks in his introduction, "We merely indicate the papers we have actually used; as a result insufficient tribute is paid to those who have founded the theory of chains, such as A. A. Markov, N. Kolmogorov, W. Doeblin, J. L. Doob and K. L. Chung."

Revuz's book is carefully organized, the proofs are well thought through and elegant. Many problems supplement the material in the text. A more extensive index, and a list of symbols would, however, be desirable. The first few chapters provide a good introduction to Markov chains for the novice, and most experts will find something new in the latter chapters.

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