sequential estimation of the mean of a normal distribution under rather general loss and cost function. Yet, it is a classical paper whose method for proving admissibility is often quoted. His work on the mean of a rectangular population is misstated on p. 342.

If I have given the impression that Govindarajulu's book only contains accounts of other people's work, with nothing added of his own, then that is not quite fair. A small amount of his own research is incorporated. The problems he supplies have been mentioned earlier. Furthermore, Govindarajulu did catch a few mistakes in the sources from which he borrowed. For instance, he points out a rather bad error in a footnote on p. 72. On p. 214 he points out an error in B. K. Ghosh's book which invalidates Ghosh's argument. Unfortunately, when trying to correct the computation and the argument he also commits an error (in the differentiation of the function h) and so his expression for $h'(\alpha)$ is incorrect. (It is claimed that $h'' < 0$ in the interval $(0, \frac{1}{2})$, but in fact $h''(\alpha) > 0$ for α close to $\frac{1}{2}$. I have no doubt that $h > 0$ in $(0, \frac{1}{2})$, but I have not seen a proof yet.)

Theorems 2.4.2, 2.4.3, and 2.4.4 seem to be new (except, of course, part (i) of Theorem 2.4.2). Unfortunately, the assumptions are not completely stated. Worse is that no proofs are supplied. There is a lot of manipulation, which is valid provided the interchange of summation and expectation can be justified, but Govindarajulu never provides this justification. Dominated convergence does not seem to work. I am much obliged to Professor Tze Leung Lai for showing me a martingale proof of Theorem 2.4.2(ii). So at least that result seems to be true, even though not proved in the book. Another question is what the formulas in Theorems 2.4.2–2.4.4 are good for. In a remark on p. 36 Govindarajulu says that in the case of Wald's SPRT Theorem 2.4.2 leads to approximate expressions for EN and Var N. But all formulas (also the ones in Theorems 2.4.3 and 2.4.4) contain only the first moment of N, so it is hard to see what Var N would follow from. Govindarajulu continues on p. 36 with some more puzzling remarks about the conditional expectation of N given that the hypothesis is accepted (or rejected). How (2.4.15)–(2.4.17) are going to be used for that purpose is a mystery to me.

In conclusion, I would like to transmit to Professor Govindarajulu my sincere regrets that this review turned out to be so negative. But in my mind a prerequisite for reaching an audience, be it by spoken or by written word, is a deep concern with the manner in which the thoughts are going to be conveyed. In my opinion the book fails to display that kind of concern.

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Sobolev spaces, by Robert Adams, Academic Press, New York, 1975, xviii + 268 pp., \$24.50.

This monograph is devoted to the study of real valued functions u defined on an open set Ω in Euclidean n-space R^n having the property that u and all its distribution derivatives up to (and including) order m are functions that are pth power summable. Here $1 \leqslant p < \infty$ and m is a positive integer. The set of all such functions u is denoted by $W^{m,p}(\Omega)$ and when endowed with an appropriate norm, for example,

$$\|u\|_{m,p} = \left\{ \sum_{0 \leqslant |\alpha| \leqslant m} \|D^\alpha u\|_p^p \right\}^{1/p}$$

the $W^{m,p}(\Omega)$ are called Sobolev spaces.

Sobolev established important results concerning these functions in **[S1]** and then incorporated a very readable account of them in his book **[S2]**. However, there were others who recognized the importance of these functions before the appearance of Sobolev's work in 1938. As early as 1906, Beppo Levi considered continuous functions of two variables that were absolutely continuous in each variable for almost all values of the other, and whose partial derivatives were in L^2 **[LE]**. Levi's functions are contained in $W^{1,2}$. In order to facilitate his investigations on the area of a nonparametric surface, L. Tonelli, in 1926, introduced his ACT functions which were the same as Levi's except that the partial derivatives were required to be in L^1 **[T]**. It is now known that the Lebesgue area of the graph of an ACT function u is given by $\int \sqrt{(1 + |\operatorname{grad} u|^2)}\, dx$. In his investigations of potential theory, G. C. Evans employed "functions which are potential functions of their generalized derivatives," **[E1]**, **[E2]**. These are essentially the same as functions in $W^{1,2}$. At the same time that Sobolev was developing his theory, J. W. Calkin and C. B. Morrey were treating existence questions in certain variational problems by allowing as admissible candidates, functions in $W^{1,p}$ (then called $\mathfrak{P}_p$) spaces. Morrey reported on these results at an invited lecture to the American Mathematical Society on December 2, 1939 **[MO3]**. The results appeared in **[C]**, **[MO1]**, and a final installment appeared in **[MO2]**. Since that time many writers have made and continue to make important contributions to the subject with applications being found in the calculus of variations, partial differential equations approximation theory, and other areas.

In view of the subject's long history and importance to many areas of analysis, it is only fitting that someone undertake the task of writing a monograph subject to the conditions listed in the author's introduction: "The existing mathematical literature on Sobolev spaces and their generalizations is vast, and it would be neither easy nor particularly desirable to include everything that was known about such spaces between the covers of one book. An attempt has been made in this monograph to present all the core material in sufficient generality to cover most applications, to give the reader an overview of the subject that is difficult to obtain by reading research papers, and finally, as mentioned above, to provide a ready reference for someone requiring a result about Sobolev spaces for use in some application." There are many books that discuss various aspects of Sobolev spaces, cf. **[A]**, **[F]**, **[MO4]**, **[ST]**. However, the reviewer is not aware of any book that develops the subject by starting from first concepts, includes many of the important results, and presents the material in a manner that a graduate student in analysis will find accessible and not too difficult to read. It is in this sense that the book under review has filled a void that has been allowed to remain for too long a time. But on the other hand, while it is true the author has presented many fundamental results of Sobolev spaces, this reviewer believes that by not including enough material which is central to the subject, the author has failed to reach the objective as stated in his introduction. The main thrust of this monograph is in the direction of imbedding theorems, including a detailed analysis of the Sobolev imbedding theorem and of the Rellich-Kondrachov theorem. However, excluded from the development are considerations pertaining to the continuity and differentiability properties of Sobolev functions, relationships between weak and pointwise derivatives, pointwise behavior of Sobolev functions, a discussion of the capacity which provides the appropriate context for pointwise behavior, and a treatment of nonintegral order Sobolev spaces that is more comprehensive than the one presented in Chapter VII. Granted that while the character and number of results which are considered vital to the development of a theory is to some extent a reflection of one's taste and point of view, it nevertheless would have benefited the reader significantly if the above topics were at least mentioned along with the appearance of the appropriate references in the bibliography.

The material in this book is organized into eight chapters, the first two of which are devoted to some of the highlights of real and functional analysis. Included here are remarks on topological vector spaces, the weak topology, operators and imbeddings, distributions and weak derivatives, the uniform convexity of L^p, the dual of L^p, and approximation by smooth functions.

Chapters III-VI form the main part of the monograph. Chapter III is entitled "The spaces $W^{m,p}(\Omega)$" and includes a development of the basic properties of integer order Sobolev spaces. The spaces $W^{-m,p'}(\Omega)$ and duality are discussed along with the fundamental result of Meyers and Serrin [**MS**] which states that $W^{m,p}(\Omega)$ is precisely the closure in the Sobolev norm of functions in $C^m(\Omega)$. Finally, the concept of (m, p) polarity is discussed, a notion which was first introduced by Hörmander and Lions [**HL**] and used by many for the purpose of treating removable singularities, see also [**AP**], [**LI**], [**SE1**], and [**SE2**]. Closely associated with (m,p) polarity is the notion of (m,p)-capacity, a concept introduced and developed independently by Havin and Maz'ya [**HM**], Meyers [**ME**], and Rešetnjak [**R**]. It is unfortunate that the author did not discuss or even give reference to some of the basic results of this capacity theory, for they contribute to a much better understanding of polarity and the behavior of functions in $W^{m,p}$. Another item that the author might have included in this chapter is the result which states that $u \in W^{1,p}(R^n)$ if and only if (i) $u \in L^p(R^n)$, (ii) that u can be modified on a set of measure zero so that it is absolutely continuous on almost all lines parallel to the coordinate axes, and (iii) the gradient $\nabla u \in L^p(R^n)$. As mentioned above, this contains, as special cases, the characterizations of Sobolev functions that Levi and Tonelli used, thus giving the result a special historical importance. Also, the result is useful, especially so when it is recalled that $u \in W^{m,p}$ if and only if u and each of its partial derivatives belong to $W^{m-1,p}$.

Chapter IV deals with interpolation and extension theorems and includes such fundamental inequalities as

$$|u|_{j,p} \leqslant K\varepsilon|u|_{m,p} + K\varepsilon^{-j/(m-j)}|u|_{0,p}$$

for any $u \in W^{m,p}(\Omega)$ where Ω is a suitably regular domain. Here, $0 \leqslant j \leqslant m-1$ and

$$|u|_{j,p} = \left\{ \sum_{|\alpha|=j} \int_\Omega |D^\alpha u|^p \right\}^{1/p}.$$

For certain bounded domains $\Omega \subset R^n$, there exists a linear operator E mapping functions u on Ω to functions on R^n with the properties:

(i) the restriction of $E(u)$ to Ω is u,

(ii) E maps $W^{m,p}(\Omega)$ continuously into $W^{m,p}(R^n)$ for all $1 \leqslant p < \infty$ and all positive integers m.

The author elects to establish this theorem under stronger than necessary regularity assumptions on Ω, thus making it possible to keep the proof relatively simple. However, it could have been mentioned that the theorem remains valid if Ω is assumed only to be a Lipschitz domain.

The fifth chapter is concerned with imbedding and nonimbedding theorems. A refinement of the Sobolev imbedding theorem is presented where conditions are given for the existence of the continuous injection $W^{j+m,p}(\Omega) \to W^{j,q}(\Omega^k)$; Ω^k denotes the intersection of the domain Ω with a k-dimensional plane. A nice feature of this chapter is the inclusion of several examples showing that in certain respects, the imbedding is best possible. The chapter is concluded with a discussion on extending the imbedding theorem to certain domains with boundary irregularities comparable to standard cusps.

Whenever Ω_0 is a subdomain of Ω, the linear restriction operator $u \to u|\Omega_0$ can be composed with the imbeddings of Chapter V to yield imbeddings of the form

$$(*) \qquad W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_0^k).$$

The question of interest throughout Chapter VI is under what conditions will this operator be compact. In the event that Ω satisfies the condition of the Sobolev imbedding theorem and Ω_0 is bounded, then the Rellich-Kondrachov theorem states that the imbedding $(*)$ is compact. The problem of whether this result can be extended to unbounded domains is considered and the author includes his own results in this direction. He characterizes those domains Ω for which the imbedding $W_0^{j+m,p}(\Omega) \to W^{j,p}(\Omega^k)$ is compact, where $W_0^{j+m,p}(\Omega)$ denotes the closure (in the Sobolev norm) of smooth functions whose supports are contained in Ω. The characterization is given in terms of a certain capacity which is different from the one developed in **[HM]**, **[ME]** and **[R]**. The final portion of the chapter is devoted to showing that if the domain's n-dimensional measure decreases rapidly near infinity, then $W^{m,p}(\Omega) \to L^p(\Omega)$ is compact.

In this and the previous chapter, the author proves the simplest versions of the Sobolev and Poincaré inequalities for smooth functions with compact support. However, a treatment (or reference) of similar inequalities for Sobolev functions with noncompact support is missing, which is unfortunate, for these inequalities are fundamental and have many important applications, cf., **[DG]**, **[D]**, **[MA]**, **[P]**, **[LU]**.

There are many different methods of extending the notion of Sobolev space to permit nonintegral values of m. These various methods do not, in general, lead to the same family of spaces. In practice, the method and its resulting spaces of functions to be used depends upon the application under consideration. In Chapter VII, the author decides to develop the spaces $W^{s,p}(\Omega)$ by means of the trace interpolation method of Lions for the purpose of providing a characterization of traces of functions in $W^{m,p}(\Omega)$ on smooth manifolds. The final pages of the chapter are given to a brief overview of other fractional spaces, namely, spaces of Bessel potentials, Besov spaces, and the spaces of Nikol'skiĭ.

The final chapter deals with Orlicz and Orlicz-Sobolev spaces, the latter being those functions whose distribution derivatives belong to an Orlicz space. In the case $mp = n$ and $p > 1$, the Sobolev imbedding theorem yields, for suitably regular domains Ω, $W^{m,p}(\Omega) \to L^q(\Omega)$ for any $p \leqslant q < \infty$. However, $W^{m,p}(\Omega)$ is not contained in $L^\infty(\Omega)$. An interesting result of Trudinger [**TR**] is given where an optimal imbedding can be found in terms of an Orlicz space. The final main result is due to Donaldson and Trudinger [**DT**] which is concerned with imbeddings of Orlicz-Sobolev spaces.

In conclusion, the author has succeeded in presenting a well-written and accessible account of much of the basic material concerning Sobolev spaces. The researcher who is interested in pursuing the subject beyond its first stages will find this monograph lacking in some respects; but for those who are looking for a source that will provide a first exposure to the theory of Sobolev spaces, this volume represents a welcome addition to the literature.

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Mathematical statistical mechanics, by Colin J. Thompson, The Macmillian Company, New York, 1972, ix + 278 pp., $15.95.

Statistical mechanics and mathematical problems, Battelle, Seattle, 1971 Rencontres, Edited by A. Lenard, Springer-Verlag, Berlin, Heidelberg, New York, 1973, v + 246 pp., $9.90.

Phase transitions and critical phenomena, Edited by C. Domb and M. A. Green, Academic Press, London, New York, Vol. 1, *Exact results*, 1972, xv + 506 pp., $31.00. Vol. 2, 1972, xv + 518 pp., $31.00. Vol. 3, *Series expansion for lattice models*, 1974, xviii + 694 pp., $46.50.

Among the major scientific issues having a mathematical component, there are a number for which the process of quantification and model building is incomplete (e.g. economics, genetics and ecology). In contrast are issues which can be clearly formulated (but not yet solved) in mathematical terms. Mathematical physics has a particularly rich collection of this second class of problems; we mention the instabilities of plasmas, the singularities of space time allowed by Einstein's equations for general relativity, turbulence, the renormalization of quantum fields, and the theory of critical behavior in statistical mechanics. The importance of these problems to physics is clear. Their importance to mathematics lies in the expectation that their solution will require new developments in–or perhaps even new branches of–mathematics.