BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, July 1976

INFINITE DIMENSIONAL VERSIONS OF TWO THEOREMS OF CARL SIEGEL[1]

BY BUCK WARE

Communicated by John Kelley, March 8, 1976

1. **The center theorem.** Let U be an open neighborhood of the origin in a Banach space E and f: $U \longrightarrow E$ an analytic function with constant term $f(0) = 0$ and linear term $u = (Df)_0$ a toplinear isomorphism.

In 1871 Ernst Schröder [4], motivated by problems in the iteration of functions, considered the problem of when f is analytically conjugate to its linear part u, that is, when does there exist an analytic function φ with $\varphi(0) = 0$ and $(D\varphi)_0 = \mathrm{id}_E$ such that $\varphi^{\leftarrow} \circ f \circ \varphi = u$. Jules Farkas [2] and Gabriel Koenigs [3] independently settled this question for contractions, $|u| < 1$, in one complex dimension by showing in 1884 that there always exists an analytic Schröder series.

The center case, $|u| = 1$, remained an enigma for fifty-eight years after the work of Farkas and Koenigs, although many people attacked the problem in the 'twenties' and 'thirties'. Let

$$(\mathrm{id} - u_*^{-1}u^*)_k\colon \mathcal{P}_k(E, E) \longrightarrow \mathcal{P}_k(E, E)$$

be defined on k-homogeneous polynomials p on E with coefficients in E by

$$(\mathrm{id} - u_*^{-1}u^*)_k\colon p \longmapsto p - u^{-1} \cdot p \circ u.$$

In 1942 Carl Siegel [5] introduced his multiplicative small divisor diophantine inequalities: say that u is "non-Liouville" iff there are constants c, ν such that, for each $k > 2$, $(\mathrm{id} - u_*^{-1}u^*)_k$ is an isomorphism and $|(\mathrm{id} - u_*^{-1}u^*)_k^{-1}| < ck^\nu$. In his celebrated six page paper Siegel proved in the one dimensional complex case the Siegel center theorem: *If u is non-Liouville then f is analytically conjugate to u.* The proof is by the majorant method and depends on delicate number-theoretical estimates. Shlomo Sternberg [7] extended the center theorem to the finite dimensional case in 1961.

Meanwhile several people, including John Nash, Andrei Kolmogorov, Vladimir Arnol'd, and Jürgen Moser, had developed a general technique for handling

---

AMS (MOS) subject classifications (1970). Primary 34C20, 39A30; Secondary 30A20.
Key words and phrases. Analytic linearization, small divisors, center theorem, Schröder's equation, normal form.

[1]Most of the results announced comprise part of a thesis written at the University of California, Santa Cruz under the direction of Al Kelley.

small divisor difficulties, the accelerated convergence method. This method has been successfully applied to obtain new theorems on mappings and hamiltonian mechanics as well to recover the results of Carl Siegel. In 1972 Stephen Diliberto [1] announced a new technique for attacking small divisor problems. We have used the Diliberto method of bounded dominants in [8] to obtain a restricted version of the center theorem in Hilbert space.

THEOREM. *Suppose that u is a center-stable automorphism, $|u| \leqslant 1$, of the Hilbert space E. If u is non-Liouville, then f is analytically conjugate to u.*

The idea of the Diliberto procedure is to recursively compute polynomial maps φ_k such that the analytic function $f_{k+1} = \overleftarrow{\varphi_k} \circ f \circ \varphi_k - u$ has order $\geqslant k + 1$, and then to keep track of the growth and radius of convergence of f_{k+1} in such a way as to ensure that the limit of the φ_k's defines an analytic function.

2. **The linearization theorem.** Let U be an open neighborhood of the origin in a Banach space E and ξ: $U \longrightarrow E$ an analytic function with constant term $\xi(0) = 0$ and linear term (Hessian) $v = (D\xi)_0$. We regard ξ as a vectorfield on U, so the derivative v of ξ at the stationary point 0 is a logarithm of the derivative $u = (Df)_0$ of the time-one map f of the flow of ξ at the fixed point 0, that is, $u = e^v$.

When working with maps Siegel considered how close the operator $u_*^{-1} u_k^*$ gets to the identity, so for vectorfields it is reasonable to take logarithms and look at

$$\log(\mathrm{id}) - \log(u_*^{-1} u_k^*) = \log(u_*) - \log(u_k^*).$$

The logarithm of u_* is just v_*, and the logarithm of u_k^* turns out to be the endomorphism $\theta_k(v)$ of $P_k(E, E)$ defined by $\theta_k(v)$: $p \longmapsto (Dp) \cdot v$; here $(Dp) \cdot v$ is the polynomial map defined by $(Dp) \cdot v$: $x \longmapsto (Dp)_x \cdot v(x)$ for $x \in E$. In 1952 Carl Siegel [6] introduced his additive small divisor diophantine inequalities: say that v is "non-Liouville" in case there exist constants c, ν such that, for $k > 2$, $v_* - \theta_k(v)$ is invertible and $|(v_* - \theta_k(v))^{-1}| < ck^\nu$. He then proved the Siegel linearization theorem for finite dimensional vectorfields: If v is non-Liouville then ξ is analytically equivalent to v.

Siegel's proof was again by a majorant argument, but of course the theorem has subsequently been proved many times by the rapid convergence method. We have used the Diliberto procedure to extend the linearization theorem to any Banach space.

REFERENCES

1. Stephen P. Diliberto, *A new technique for proving the existence of analytic functions*, Ordinary Differential Equations: 1971 NRL–MRC Conference (edited by Leonard Weiss), Academic Press, New York, 1972, pp. 71–81. MR 50 #5065.

2. Jules Farkas, *Sur les functions itèratives*, J. Math. Pures Appl. (3) 10 (1884), 101–108.

3. Gabriel P. X. Koenigs, *Recherches sur les intégrales des certaines équations fonctionnelles*, Ann. Sci. École Norm. Sup. (3) **1** (1884), Supplement 3–41.

4. F.W.K. Ernst Schröder, *Über iterierte Funktionen*, Math. Ann. **3** (1971), 296–322

5. Carl L. Siegel, *Iteration of analytic functions,* Ann. of Math. (2) **43** (1942), 607–612. MR **4**, 76.

6. ———, *Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung*, Nachr. Akad. Wiss. Göttingen. Math. -Phys. Kl. Math. -Phys. -Chem. Abt. **1952**, 21–30. MR **15**, 222.

7. Shlomo Sternberg, *Infinite Lie groups and the formal aspects of dynamical systems,* J. Math. Mech. **10** (1961), 451–474. MR **24** #A3234.

8. Buck Ware, *The Siegel centerstable theorem in Hilbert space*, Bericht no. 37, Mathematischstatistischen Sektion im Forschungszentrum, Graz., 1975

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA CRUZ, CALIFORNIA 95064