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DEGENERATION AND SPECIALIZATION IN ALGEBRAIC FAMILIES OF VECTOR BUNDLES

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Introduction. If X is a nonsingular d-dimensional projective variety (over an algebraically closed field, k), then by a *vector bundle* on X we understand a torsion-free, coherent $\mathcal{O}_X$-module. The classical notion of vector bundle (= locally free $\mathcal{O}_X$-module) is subsumed in the new definition and agrees with it when X is a curve. The new notion is necessary for the study of moduli problems [G], [L] and because the classical definition is too rigid for higher dimensional varieties [L]. However, given a vector bundle on X, there exists an open subset U of X containing all points of codimension $\leqslant 1$ such that the vector bundle is locally free over U. Chern classes can always be defined for such bundles.

We fix, once and for all, a very ample divisor class, H, on X. Then for any prescheme S over k, the line bundle p^*H on $X \times_k S$ (where $p\colon\ X \times_k S \longrightarrow X$ is the projection) is very ample with respect to S. As usual, set $F(n) = F \otimes H^{\otimes n}$, then the Hilbert polynomial $p_F(n)$ is defined and, by the Riemann-Roch Theorem, $p_F(n)$ has an expression in terms of Chern classes. Indeed, if we set

$$\deg F = (c_1(F) \cdot H^{d-1}) \quad \text{(intersection no.)},$$

and refer to $\deg F$ as the *degree of F* (more properly, the H-degree of F), then

$$\frac{p_F(n)}{\operatorname{rk} F} = \delta(X)\frac{n^d}{d!} + \left(\frac{\deg F}{\operatorname{rk} F} - \frac{\deg K}{2}\right)\frac{n^{d-1}}{(d-1)!}$$
$$+ \text{ terms of lower degree in } n.$$

The quantities $\widetilde{\mu}(F) = p_F(n)/\operatorname{rk} F$, $\mu(F) = (\deg F)/\operatorname{rk} F$ are fundamental for what we shall do. The former was introduced by Gieseker, the latter by Takemoto [T], and both were inspired by results of Mumford for curves. We shall concentrate on μ; however, everything we do carries over to $\widetilde{\mu}$ suitably interpreted. Call $\mu(F)$ the *slope of F*.

The bundle F *is semistable* (resp. *stable*) if for every proper subbundle, G, $\mu(G) \leqslant \mu(F)$ (resp. $\mu(G) < \mu(F)$). Our bundle F is *unstable* if it is not semistable. We regard an unstable bundle as "more degenerate" than a semistable bundle, *etc.*

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Main results. Let F be a bundle on X, as above. A *Harder-Narasimhan Flag for F* (abbreviated HNF(F)) is a descending chain of subbundles of F,

$$(*) \qquad F > F_{t-1} > \cdots > F_1 > (0),$$

having the properties:

(a) each factor bundle F_j/F_{j-1} is semistable;

(b) $\mu(F_{j+1}/F_j) < \mu(F_j/F_{j-1})$, $1 \leqslant j \leqslant t-1$.

(These chains were introduced in [**HN**] in the case X was a curve, for other purposes.)

THEOREM 1. *Every bundle, F, on the d-dimensional, nonsingular, irreducible, projective variety, X, possesses a unique HNF. Any two flags* $(*)$ *satisfying* (a) *and* (b) *are identical. Moreover, if F is itself semistable, then F possesses a flag* $(*)$ *in which*

(a) *each factor bundle is stable, and*

(b) $\mu(F_{j+1}/F_j) = \mu(F_j/F_{j-1})$.

Theorem 1 is not difficult to prove if one makes full use of the new definition of vector bundle–it is false otherwise. Given a bundle F, if we plot in the (rk, deg)-plane the points whose coordinates are the ranks and degrees of the bundles occurring in $(*)$ for F, we obtain a polygon which we call the *Harder-Narasimhan Polygon for F* (HNP(F)). The slopes of the sides of this polygon are exactly the numbers $\mu(F_1)$, $\mu(F_2/F_1)$, *etc.* ocurring in (b) above. Theorem 1 states that every F possesses a unique HNP, and that HNP(F) *is a convex polygon.*

Let X be a nonsingular, irreducible, projective k-variety and let S be a scheme over k. A vector bundle on $X \times_k S$, *flat over S*, will be called an *algebraic family of vector bundles on X parametrized by S.* If F is a family on X and $s \in S$, we let F_s denote the pull-back of F to the fibre, X_s, of $X \times_k S$ over s. The divisor class H on X induces divisor classes p^*H and H_s on $X \times_k S$ and X_s, respectively, by pull-back. These are very ample as H is, and semistability is measured *via* these given very ample sheaves.

THEOREM 2.[2] *Given X, S, and F on $X \times_k S$, as above, form HNP(F_s) for each $s \in S$. If $t_0 \in S$ is a specialization of $s \in S$, then HNP(F_{t_0}) lies on or above HNP(F_s). That is, the Harder-Narasimhan polygon rises under specialization.*

When the moduli scheme for stable bundles exists (i.e., for X a curve [**M**] or a surface [**G**]), *one can show that* HNP(F_s) *is a constructible function of s,* therefore Theorem 2 implies it is *upper semicontinuous*. Using the upper semi-

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continuity, we have obtained a fundamental map from vector bundles on $X \times_k S$ to collections of $r - 1$ algebraic cycles (with nonnegative coefficients) of S, r being the rank of the bundles.

In the simplest case, let Y be a ruled surface with base curve C, and let Y_c denote the generic fibre. Fix a convex polygon, P, with vertices at (0, 0), $(1, d_1), \ldots, (r, d_r)$ (the d's are given integers), and let Vect(Y; P) denote the set map of rank r bundles on Y whose HNP at the generic fibre Y_c is the given polygon P. Then we obtain a

$$\operatorname{Vect}(Y; P) \xrightarrow{\theta} \prod_{r-1 \text{ factors}} \operatorname{Hilb}(C).$$

It turns out that two bundles on Y have the same image under θ if and only if their pull-backs to *each* fibre of Y over C are *isomorphic*, moreover the map θ is surjective. These matters are discussed in **[S₁]**, **[S₂]**, **[S₃]**.

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