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THE PRINCIPAL SYMBOL OF A DISTRIBUTION

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In Hörmander's theory of Fourier integral operators [1], a principal symbol is constructed for a certain class of distributions in such a way that, when the construction is applied to the Schwartz kernel of a pseudodifferential operator, one obtains the usual principal symbol of the operator. In this note, we describe a generalization of Hörmander's construction which may be applied to an arbitrary distribution on a manifold. Details will appear in [4].

1. **Local definition and invariance properties.** For a complex vector space V, we define V-valued distributions on $\mathbf{R}^n$ by taking as test functions objects of the form $u = u(x)\,dx$, where $u(x)$ is a compactly supported C^∞ function with values in V^*, and dx is the density $|dx_1 \wedge \cdots \wedge dx_n|$. For $\tau > 0$, we define u_τ to be $u(\tau x)\,dx$. If g is a V-valued distribution, and φ is a C^∞ function with $\varphi(0) = 0$, we define the family $\{g_\tau^\varphi\}_{\tau>0}$ of distributions by

$$\langle g_\tau^\varphi, u\rangle = \langle g, e^{-i\tau\varphi} u_{\sqrt{\tau}}\rangle. \tag{1}$$

For $N \in \mathbf{R}$, we write $g_\tau^\varphi \in O(\tau^N)$ if $\tau^{-N} g_\tau^\varphi$ remains bounded in distribution space [3] as $\tau \longrightarrow \infty$.

LEMMA. *For every g and φ, $g_\tau^\varphi \in O(\tau^N)$ for some $N \in \mathbf{R}$.*

DEFINITION. $\inf\{N \mid g_\tau^\varphi \in O(\tau^N)\} \in [-\infty, \infty)$ is called *the order of g at φ* and denoted by $O_\varphi(g)$.[2]

THEOREM 1. (a) *If $O_\varphi(g) \leqslant N$ and $\psi(x) = \varphi(x) + \Sigma a_{jk}x_jx_k + O(x^3)$, then $g_\tau^\psi - e^{-i\Sigma a_{jk}x_jx_k} g_\tau^\varphi \in O(\tau^{N-1/2})$.*

(b) *If $O_\varphi(g) \leqslant N$ and A is a C^∞ function with values in* Hom(V, V), *then $(Ag)_t^\varphi - A(0)g_\tau^\varphi \in O(\tau^{N-1/2})$.*

(c) *If $O_\varphi(g) \leqslant N$ and $\theta\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a diffeomorphism with $\theta(0) = 0$, then $(\theta^* g)_\tau^{\theta^*\varphi} - (T_0\theta)^*(g_\tau^\varphi) \in O(\tau^{N-1/2})$.*

DEFINITION. If $O_\varphi(g) \leqslant N$, the class of $\tau^{-N} g_\tau^\varphi$ modulo $O(\tau^{-1/2})$ is called the *principal symbol* of order N for g at φ.

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[2] See final note added in proof.

2. **Global theory.** The space $\mathcal{D}'(X; E)$ of distributions on a manifold X with values in a vector bundle E is defined as the dual of the space of compactly supported C^∞ sections of $E^* \otimes \Omega_X$. (Ω_X = densities on X.) Using local coordinates on X and local trivializations of E, and applying Theorem 1, we can define for each $(x, \xi) \in T^*X$ the order $O_{(x,\xi)}(g)$ of g at (x, ξ).

To define the principal symbol in an invariant way, we must take into account the effect of changes in φ and coordinate changes as given by Theorem 1. For each $(x, \xi) \in T^*X$, the 2-jets of functions $\varphi \in C^\infty(X; \mathbf{R})$ with $\varphi(x) = 0$ and $d\varphi(x) = \xi$ can be identified with the elements of the space $L_{(x,\xi)}$ of lagrangian subspaces in $T_{(x,\xi)}T^*X$ transversal to the fibre. The additive group Q_x of homogeneous quadratic functions on T_xX acts simply and transitively on $L_{(x,\xi)}$, and it also acts on $\mathcal{D}'(T_xX; E_x)$ by $(a, g) \mapsto e^{-ia}g$. The space $\mathcal{U}_{(x,\xi)}(X; E)$ is defined to consist of the Q_x-equivariant maps from $L_{(x,\xi)}$ to $\mathcal{D}'(T_xX; E_x)$, and $S^N_{(x,\xi)}(X; E)$ is defined as the space of families $\{g_\tau\}_{\tau>0}$ in $\mathcal{U}_{(x,\xi)}(X; E)$ with $g_\tau \in O(\tau^N)$.

Now, if $O_{(x,\xi)}(g) \leqslant N$, Theorem 1 implies that the principal symbol $\sigma^N_{(x,\xi)}(g)$ of order N for g at (x, ξ) is well defined as an element of $S^0_{(x,\xi)}(X; E)/S^{-1/2}_{(x,\xi)}(X; E)$. If $\sigma^N_{(x,\xi)}(g)$ is of the form $g_0 + S^{-1/2}_{(x,\xi)}(X; E)$, where g_0 is a constant in $\mathcal{U}_{(x,\xi)}(X; E)$, we say that g is homogeneous of order N at (x, ξ) and, by abuse of notation, write $\sigma^N_{(x,\xi)}(g) = g_0$. If $O_{(x,\xi)}(g) = N$, we simply call $\sigma^N_{(x,\xi)}(g)$ the principal symbol of g at (x, ξ) and denote it by $\sigma_{(x,\xi)}(g)$. (If $O_{(x,\xi)}(g) = -\infty$, we define $\sigma_{(x,\xi)}(g)$ to be zero.)

As a first step toward a general calculus of principal symbols, we have:

THEOREM 2. *Let P be a pseudodifferential operator of order k and type $(1, 0)$ from E to F with homogeneous principal symbol $(x, \xi) \longrightarrow p(x, \xi) \in$* $\mathrm{Hom}(E_x, F_x)$. *If $O_{(x,\xi)}(g) \leqslant N$, $\xi \neq 0$, then $O_{(x,\xi)}(Pg) \leqslant N + k$, and $\sigma^{N+k}_{(x,\xi)}(Pg) = p(x, \xi)\sigma^N_{(x,\xi)}(g)$.*

COROLLARY. $(x, \xi) \notin WF(g) \Rightarrow O_{(x,\xi)}(g) = -\infty$.

3. **The principal symbol of a Fourier integral distribution.**

DEFINITION. If S is a subspace of T_xX, an *E_x-valued δ-function on S* is a distribution $\delta \in \mathcal{D}'(T_xX; E_x)$ such that:

(a) δ is continuous for the C^0 topology;

(b) δ is supported on S;

(c) δ is translation-invariant by S.

For example, the δ-densities on S correspond to the translation invariant measures on S and form a 1-dimensional space.

Now let $K \subseteq T_{(x,\xi)}T^*X$ be a lagrangian subspace, with projection $\overline{K}$ in T_xX.

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[3] This construction, as well as that of $\mathcal{U}_{(x,\xi)}(X; E)$ itself, is closely related to the symplectic spinors of Kostant and Sternberg [2].

For each E_x-valued δ-function δ on $\overline{K}$ there is a unique $\delta_K \in \mathcal{U}_{(x,\xi)}(X; E)$ which assigns δ to every $L \in \mathsf{L}_{(x,\xi)}$ such that $\dim(L \cap K) = \dim K$. The set of all δ_K, as δ runs over the E_x-valued δ-functions on $\overline{K}$, is a $\dim E_x$ dimensional subspace $\Delta_K(E)$ of $\mathcal{U}_{(x,\xi)}(X; E)$.[3]

THEOREM 3. *Let $g \in \mathcal{D}'(X; E)$ be a Fourier integral distribution associated with the conic lagrangian submanifold $\Lambda \subset T^*X$ and having order m, type* $(1, 0)$ *and homogeneous principal symbol. Let $(x, \xi) \in \Lambda$, $K = T_{(x,\xi)}\Lambda$.*

(a) *g is homogeneous of order $\overline{m} = m + \frac{1}{4}\dim X$ at (x, ξ).*

(b) $\sigma^{\overline{m}}_{(x,\xi)}(g) \in \Delta_K(E)$.

(c) *If E is the bundle of ½ densities on X, then $\Delta_K(E)$ is naturally isomorphic with the fibre over (x, ξ) of the symbol bundle $\Omega_{1/2} \otimes L$ of* [1], *and $\sigma^{\overline{m}}_{(x,\xi)}(g)$ is equal to the principal symbol as given in* [1].

REMARK. We can show directly that $\Delta_K(E)$ depends smoothly on lagrangian $K \subset T_{(x,\xi)}(T^*X)$, thus giving a new, analytic, construction of the symbol bundle.

ADDED IN PROOF (JUNE 2, 1976). As A. Douady has pointed out to me, the set $\{N \mid g^{\varphi}_{\tau} \in O(\tau^N)\}$ could be an interval of the form (a, ∞). In this case, we should define $O(\tau^N)$ to be a^+, where a^+ lies by convention between a and any number greated than a.

REFERENCES

1. L. Hörmander, *Fourier integral operators.* I. Acta Mathematica **127** (1971), 79–183.

2. B. Kostant, *Symplectic spinors,* Symposia Mathematica **14** (1974).

3. L. Schwartz, *Théorie des distributions.* Nouvelle rev. ed., Hermann, Paris, 1966. MR **35** #730.

4. A. Weinstein, *The order and symbol of a distribution,* Inst. Hautes Etudes Sci. Publ. Math., 1976, (preprint).

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