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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, July 1976

Commutative formal groups, by Michel Lazard, Lecture Notes in Mathematics, vol. 443, Springer-Verlag, Berlin, Heidelberg, New York, 1975, 236 pp., \$9.90.

This is the book we have been waiting for ever since P. Cartier's pair of notes in the Comptes Rendus of 1967. In these, Cartier sketched a thoroughgoing extension of the Dieudonné theory that had already classified commutative formal groups over a perfect field of characteristic p, in terms of modules over a certain noncommutative ring. But Cartier left the job of exposition unfinished, and Lazard has done us the service of organizing the material, filling in all the details, and adding a quantity of his own results, so that we finally have a basic reference on this aspect, probably the central aspect, of the theory of commutative formal groups.

An n-dimensional (coordinatized) formal group is simply an n-tuple $\mathbf{F} = (F_1, \ldots, F_n)$ of formal power series, subject to a single condition expressing a kind of associativity. Here, $F_i = F_i(\mathbf{x}, \mathbf{y})$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$; and $x_1, \ldots, y_n$ are $2n$ independent indeterminates. For instance, the expansion at the origin of the group law of an n-dimensional complex analytic Lie group gives rise to such series, once a coordinate system is chosen; the standard coordinatization of the one-dimensional multiplicative Lie group $\mathbf{C}^*$, for example, gives the single power series $F(x,y) = x + y + xy$.

The advantage in talking about formal groups rather than local groups is that the single relation of associativity $\mathbf{F}(\mathbf{F}(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathbf{F}(\mathbf{x}, \mathbf{F}(\mathbf{y}, \mathbf{z}))$ makes sense algebraically, in the ring of formal power series $A[[\mathbf{x}, \mathbf{y}, \mathbf{z}]]$, where A is any commutative ring whatever. We need not restrict ourselves to the groundrings $\mathbf{C}$ and $\mathbf{R}$, not even to topological rings, and can now ask the relationship between Lie algebras over the ring A and formal groups over A. We then find that if A is a $\mathbf{Q}$-algebra, i.e. if every positive integer is invertible in A, then the categories of finite-dimensional Lie algebras over A and of finite-dimensional formal groups over A are equivalent: in particular, two formal groups have isomorphic Lie algebras if and only if they become equal after a formal transformation of coordinates. In particular, for each $n \geqslant 0$, there is only one isomorphism class of n-dimensional commutative formal groups over such a ring A. It must have come as a great disappointment to the algebraizers of algebraic geometry, to realize that the Campbell-Hausdorff formula is next to useless when the groundring A is more general, in particular when A is a field of characteristic $p > 0$. In fact, over such a field, the Lie algebra gives information only about what might be called "the first infinitesimal group neighborhood" of the identity. It must have seemed a Herculean problem to classify merely the commutative formal groups over fields of positive characteristic; even the two simplest ones, namely the "additive" $x + y$ and the "multiplicative" $x + y + xy$ are nonisomorphic, as a glance at the endomorphisms corresponding to the integer p shows: in the additive case this is $x + \cdots + x$ (p times), and thus zero, while in the multiplicative it is $(1 + x)^p - 1 = x^p \neq 0$. This was the situation before 1954; in that year the first of Dieudonné's wideranging and formidable papers on formal groups appeared, and the picture began to change completely. By 1958, when the eighth of Dieudonné's papers was published, the subject of formal groups over perfect fields of characteristic p was well charted, and in principle well understood. In this account of the prehistory of the subject, I do not mean to have given the impression that the importance of formal groups was recognized before Dieudonné, and that the mathematical world was simply waiting for someone to shoulder the burden of analyzing their properties. In fact, although it was Bochner who first gave the basic definition, it is Dieudonné to whom we are indebted for his recognition of the significance of formal groups in algebraic geometry; it was he who created the study of formal groups out of nothing and who gave us what for a long time was the only extensive published treatment of the fundamentals of the field.

Dieudonné's method was to use "semiderivations" which would catch the infinitesimal information lost by the ordinary derivations, and "Lie hyperalgebras". It was an effective but cumbersome machinery, leading to an equivalence of categories between the finite-dimensional commutative formal groups $\mathbf{G}$ over a perfect field k of characteristic p, and certain modules $D(\mathbf{G})$ over a noncommutative ring $E(k)$ of Krull dimension two, which in turn is an algebra over $W_\infty(k)$, the ring of infinite k-valued Witt vectors. But even after conceptualization and simplification by Gabriel, there still remained no inkling of how to extend the results to the case where the groundring was other than a perfect field. One reason is that the Dieudonné-Gabriel theory uses in a very strong way the fact that a formal group $\mathbf{G}$ may be viewed as the inductive limit of the finite subgroupschemes $\ker(\rho^n)$, where ρ^n is the nth Frobenius morphism, from $\mathbf{G}$ to the formal group gotten from $\mathbf{G}$ by raising all coefficients to the p^nth power. When the groundring is of characteristic p, then, a formal group is an "indfinite object", and one could then apply the known facts about finite commutative groupschemes over a perfect field of characteristic p, in particular that the category of all such is abelian. There is, however, no hope of using this method to get Dieudonné modules $D(\mathbf{G})$ for commutative formal groups $\mathbf{G}$ over arbitrary rings: in general, $\mathbf{G}$ will not be indfinite, and even where special formal groups $\mathbf{G}$ over special groundrings A are indfinite, the finite commutative groupschemes over A will not form an abelian category.

Gabriel's form of the Dieudonné theory in fact classifies more than the formal groups described above: it works for any inductive limit of finite groupschemes (always over a perfect groundfield k). But Cartier's method returns to formal groups proper, using in a very strong way the fact that a formal group is the formal spectrum of a power series ring.

Any formal group over a ring A gives us an infinity of honest groups. Indeed, suppose $\mathbf{F}(\mathbf{x}, \mathbf{y})$ is of dimension n, over A. Let B be any commutative A-algebra which is complete (and separated) with respect to the topology defined by the powers I^n of an ideal I of B. (For instance, $B = A[[t_1, \ldots, t_r]]$, power series in r indeterminates, and $I = (t_1, \ldots, t_r)$. Another example: B is any noetherian A-algebra, and I is the ideal of all nilpotent elements; in this case, the I-topology is discrete.) Then the set of n-tuples of elements of I becomes a group under the law $\mathbf{F}$: $\mathbf{a}^* \mathbf{b} = \mathbf{F}(\mathbf{a}, \mathbf{b})$. I believe that Cartier was the first, at least as early as 1962, to observe that in this way, a formal group is much more group than formal. In any event, he took the idea and used it brilliantly to form the Cartier Dieudonné modules. Just use $B = A[[t]]$, power series in one variable. Then the ideal is $I = tA[[t]]$, and the n-tuples of power series become elements in a group $C(F)$ whose composition law is given by the n-dimensional formal group $\mathbf{F}$. Cartier calls them "curves in $\mathbf{F}$", as indeed they are, when viewed from the proper angle. They form not only a group, but a left module over a universal ring which Lazard calls Cart(A), and in fact we get a fully faithful covariant functor from formal groups over A to left Cart(A) modules. And this becomes an equivalence of categories when, instead of all Cart(A) modules, one takes a particular subcategory defined by certain finiteness conditions.

With Cartier's achievement now fully documented, the study of formal groups is no longer in its infancy. In the book under review, Lazard has given us *the* basic reference on commutative formal groups, or at least on the most important part of the theory. He ignores in his treatment, as I have in the foregoing historical sketch, such topics as the applications to Abelian varieties and local classfield theory; work by Barsotti, Serre, Tate and others on p-divisible groups; and even Quillen's applications of formal groups to algebraic topology, the most striking of several descendants of Lazard's own readable and self-contained paper on one-dimensional formal groups, published twenty years ago.

A basic reference is what we have, and not a text. It is not clear to me now how many will be able to learn the subject from the beginning out of Lazard: the organization seems relentlessly linear, so that to reach the classical Dieudonné theory for formal groups over perfect fields of characteristic $p > 0$, the beginner would have to go through most of the first three quarters of the book. But those who already have some familiarity with the subject will find that Lazard's *Commutative formal groups* is an extremely useful exposition of a part of algebraic geometry, clearly and neatly presented by an expert in the field. I expect that with the publication of this book, research on formal groups, and research that depends on the use of formal groups as a tool, will accelerate noticeably.

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