3. J. Rauch and M. E. Taylor, *Penetrations into shadow regions and unique continuation properties in hyperbolic mixed problems*, Indiana Univ. Math. J. **22** (1972/73), 277–285. MR **46** #2240.

JEFFREY RAUCH

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 3, May 1976

Latin squares and their applications, by J. Dénes and A. D. Keedwell, Academic Press, New York, 1974, 547 pp., $24.50.

A *latin square* $A=[a_{ij}]$ of order n is an $n\times n$ array in which the places are occupied by elements from an n-element set and each element from the set occurs exactly once in each row and column. They are familiar objects in algebra as multiplication tables of quasigroups, in geometry as coordinate systems for nets, and in statistics where, as one of the simplest combinatorial designs, they are used extensively in the design of experiments.

This is the first book devoted entirely to latin squares. While the statistical, algebraic and geometric aspects are discussed, the major theme is the construction of orthogonal sets of latin squares. This is not surprising since much of the current interest in latin squares was stimulated by the disposal in the late 1950's of a famous conjecture of Euler's. Two $n\times n$ latin squares $A=[a_{ij}]$, $B=[b_{ij}]$ are *orthogonal* if, when B is superimposed on A, the n^2 ordered pairs (a_{ij}, b_{ij}) contain each pair exactly once. Euler's *Officers Problem* concerns the existence of a 6×6 array of 36 officers, 6 of each rank, from 6 different regiments, such that there is, in each row and in each column, exactly one officer of each rank and one officer from each regiment. This is obviously equivalent to the existence of two orthogonal latin squares of order six. Euler was able to construct a pair of orthogonal latin squares for all orders n other than $n\equiv 2 \pmod 4$ and he conjectured that for these orders no such pair exists. That Euler's conjecture is true for $n=6$ was verified by Tarry in 1900. It was not until 1958–1960 that the combined efforts of Bose, Shrikhande and Parker showed that Euler was wrong in all other cases.

At about the same time, another well-known conjecture was disposed of by Parker. Macneish in 1922 conjectured that if $n=p_1^{i_1}p_2^{i_2}\cdots p_m^{i_m}$ (p_i distinct primes) then the maximal size of a set of mutually orthogonal latin squares (m.o.l.s.) is $(\min p_j^{p_j})-1$. This conjecture is based on the construction of a set of p^i-1 m.o.l.s. from a finite field of order p^i. By a direct product construction we can obtain from t m.o.l.s. of order n_1 and t m.o.l.s. of order n_2, a set of t m.o.l.s. of order n_1n_2. Thus, for $n=p_1^{i_1}p_2^{i_2}\cdots p_m^{i_m}$, there is a set of at least $(\min p_j^{i_j})-1$ m.o.l.s. Macneish conjectured that there were exactly this many. However, Parker constructed a set of 4 m.o.l.s. of order 21. More recently, sets of 5 m.o.l.s. of order 12 have been constructed. Since we now know that there do exist sets of m.o.l.s. for all $n>6$, interest has shifted to the question of the maximal size of such sets. We know that the number tends to infinity with n but we do not even know whether there are three m.o.l.s. of order ten. We cannot have more than $n-1$ m.o.l.s. of order n and a set of this size is called a *complete set.* Such a set is equivalent to the existence of a finite plane of order n. For n a power of a prime we can construct a plane from a finite field. For other n, we have the negative information contained in the Bruck-Ryser theorem: if $n \equiv 1, 2 \pmod 4$ and n is not the sum of two squares, then there is no plane of order n. The first unknown case is $n=10$ and hence one reason for the interest in the number of m.o.l.s. of order ten.

With the exception of Chapter 10 on applications to coding theory and experimental designs, Chapters 5 through 13 deal with various aspects of orthogonality and the construction of sets of m.o.l.s. Chapter 5 proves some elementary facts about m.o.l.s. including the equivalence of the existence of a complete set of m.o.l.s. of order n and the existence of a finite plane of order n. There is also a proof of the Bruck-Ryser theorem. Chapter 6 looks at some connections between magic squares and m.o.l.s. Chapters 7, 11, 12 are concerned with specific methods of construction of sets of m.o.l.s. These include the construction of orthogonal mates of a square by rearrangements of its rows and columns and by direct product-type recursive constructions. Chapter 11 is a very thorough treatment of the end of the Euler conjecture giving all the results and constructions of Parker, Bose and Shrikhande which led to the overthrow of both the Macneish and the Euler conjectures. Block designs and orthogonal arrays play the major role in these constructions.

A 3-*net* of order n is a "partial plane" consisting of n^2 points and three pencils of n parallel lines such that through each point there is exactly one line of each pencil. For the definition of k-*net* we replace three by k. A latin square is essentially a coordinate system for a 3-net and a set of $k-2$ m.o.l.s. of order n is equivalent to a k-net. Chapter 8 takes up the detailed study of nets and projective planes. Geometric conditions on nets correspond to algebraic conditions on the corresponding quasigroup and several of these are discussed. Chapter 9 gets into some interesting connections between graphs and latin squares.

Although quasigroups and other algebraic systems occur throughout the book, most of the results about quasigroups occur in the first three chapters. Chapter 1 contains some elementary connections between properties of a latin square and properties of the corresponding quasigroup and (in a less elementary vein) the relationship between a *transversal* of a latin square and a *complete mapping* of the quasigroup. Chapter 2 continues the study of quasigroups and contains a list of quasigroup identities which includes most of those which have been studied in any detail, e.g. the Steiner identities $x^2=x$, $xy=yx$, $x \cdot xy=y$. There is no attempt to develop a systematic theory linking identities with combinatorial properties although this idea is currently proving very fruitful.

Chapter 3 is the first of several sections dealing with generalizations of latin squares, in this case latin rectangles and partial latin squares. Some recent theorems on completing partial latin squares of various types are listed but without the algebraic consequences (hopficity, residual finiteness, solvable word problem, etc.) which provided the original motivation for trying to prove such theorems. Generalizations of latin squares to latin cubes and hypercubes are treated briefly in Chapter 5.

Two chapters discuss the difficult task of counting and classifying latin squares of a given order. Chapter 4 contains a classification of latin squares of orders $\leqq 6$ in terms of *main classes.* Chapter 13 contains a discussion of various attempts, some successful, some not, to use a computer on latin square problems. These have ranged from simply generating all latin squares (or representatives of main classes) of a given order to the construction and classification of neofields, quasifields and ternary rings. Attempts to use a computer to find a pair of 10×10 orthogonal latin squares were unsuccessful and so far this has also been the case in a search for three m.o.l.s. of order ten. However, computers combined with some theory have helped, in two different approaches, in finding large sets of m.o.l.s. of order twelve.

This outline of the contents should convince anyone interested in latin squares that the book is an invaluable reference work. It contains an enormous collection of results. Perhaps the authors did not intend to do more than this and further criticism is unnecessary. But it is a pity that the primitive appeal of latin squares does not come through. This is not a book which will "turn on" anyone not already interested in latin squares and it would be very difficult to use it as an introductory textbook about latin squares. It is true that "no prior knowledge" has been assumed on the part of the reader but it would be frustrating to read the book from the beginning since the level of sophistication oscillates back and forth. Trivial results are interspersed with deep theorems and (when no proofs are given) the beginning reader has little indication of these variations. There is no feeling in reading the book of a systematic development of ideas and one gets the impression of a disjoint collection of topics. Even if the authors intended the book simply as a survey of current work in the field, one would wish for more ideas and motivations rather than long lists of results. Many of the proofs seem to be mainly transcriptions of the original work without the further insight which should be provided.

If one considers the literature available (in English) on the three areas covered in the book, namely constructions of m.o.l.s., geometric aspects of latin squares, algebraic aspects of latin squares, there is no doubt that the first is thoroughly covered here. There have been several books recently on finite geometries (if not specifically on the topic of latin squares and finite geometries) and so here again with the treatment in this book the literature is adequate. It is on the algebraic aspects that we need some good books both at the introductory and advanced levels. There is no book in English on the topics of loops, quasigroups and related algebraic structures, other than the Ergebnisse monograph by Bruck, which is rather specialized in intent. The same author's survey paper *What is a loop*? in the MAA Studies in Modern Algebra is probably the best introduction to the subject. The theory of quasigroups sits rather nicely between combinatorics and universal algebra. It is quite a lively part of algebra at present and I am sorry that the authors did not take the opportunity to provide the very much needed introduction to the general theory of loops, quasigroups and their n-dimensional generalizations.

All in all though, the authors deserve the thanks of everyone interested in this area of combinatorics for putting together this encyclopedic study. The bibliography alone is almost worth the price of the book.

TREVOR EVANS