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GENERATORS OF THE UNITARY Z/p BORDISM RING

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I. **Serendipity**. Let p be a prime number, let $\mathbf{Z}/p$ be the cyclic group of order p and let $\mathfrak{A}_*^{\mathbf{Z}/p}$ be the unitary $\mathbf{Z}/p$ bordism ring.

THEOREM 1. *$\mathfrak{A}_*^{\mathbf{Z}/p}$ is multiplicatively generated, over $\mathfrak{A}_*$, by the following*:

$\{\Gamma^m(\mathrm{pt}), m \geqslant 0\}$,
$\bigcup\{\mathbf{Z}/p\}$,
$\bigcup\{\Gamma^m(\mathbf{C}P_j^1), m \geqslant 0, (p+1)/2 \leqslant j < p-1\}$,
$\bigcup\{S_j, 1 \leqslant j \leqslant (p-1)/2\}$,
$\bigcup\{\Gamma^m(C_j), m \geqslant 0, 1 < j \leqslant (p+1)/2\}$,
$\bigcup\{\Gamma^m(\mathbf{C}P_j^n), m \geqslant 0, n \geqslant 2, 1 \leqslant j \leqslant p-1\}$.

Furthermore, this set is irredundant.

The notation is explained by the following.

(a) pt the point, with obvious $\mathbf{Z}/p$ action.

(b) $\mathbf{Z}/p$, p points with obvious $\mathbf{Z}/p$ action.

(c) $\mathbf{C}P_j^1$, $((p+1)/2 \leqslant j < p-1)$, the complex projective, line $\mathbf{C}P^1$ with $\mathbf{Z}/p$ action given by $[z_0; z_1] \longmapsto [z_0; \xi^j z_1]$ where $\xi = \exp(2\pi i/p)$.

(d) S_j, $(1 \leqslant j \leqslant (p-1/2)$, the Riemann surface of genus $(q-1)(p-1)/2$ associated to the complex function $u = (z^p - 1)^{1/q}$ where q satisfies $qj = -1 \bmod p$, $0 < q < p$. The action of $\mathbf{Z}/p$ on S_j is induced by $z \longmapsto \xi z$.

(e) C_j, $(1 < j \leqslant (p+1)/2)$, the complex projective plane $\mathbf{C}P^2$ with $\mathbf{Z}/p$ action given by $[z_0; z_1; z_2] \longmapsto [z_0; \xi z_1; \xi^j z_2]$.

(f) $\mathbf{C}P_j^n$ $(n \geqslant 2, 1 \leqslant j \leqslant p-1)$, the complex projective space $\mathbf{C}P^n$ with $\mathbf{Z}/p$ action given by $[z_0; z_1; \ldots; z_{n-1}; z_n] \longmapsto [z_0; z_1; \ldots; z_{n-1}; \xi^j z_n]$.

(g) Let M be a unitary $\mathbf{Z}/p$ manifold for which the $\mathbf{Z}/p$ action extends to a unitary S^1 action. For example, the unitary $\mathbf{Z}/p$ manifolds in (a), (c), (e) and (f) satisfy this property.

The circle S^1 acts freely on the product $M \times S^3$ by $(m, z_1, z_2) \longmapsto (tm, tz_1, tz_2)$, $t \in S^1$, where $S^3 = \{(z_1, z_2) \in \mathbf{C}^2; |z_1|^2 + |z_2|^2 = 1\}$. Let $\Gamma(M)$ denote the quotient $(M \times S^3)/S^1$, with $\mathbf{Z}/p$ action given by $[m, z_1, z_2] \longmapsto [\xi m, \xi z_1, z_2]$. Of course, this action extends to an S^1 action and we can define,

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inductively, $\Gamma^n(M)$ by $\Gamma^n(M) = \Gamma(\Gamma^{n-1}(M))$. Also we set $\Gamma^0(M)$ to be M.

II. **Applications.** Note that most of the generators of $\mathfrak{A}_*^{\mathbf{Z}/p}$ are in fact S^1 manifolds and so we can use known results concerning S^1 manifolds to obtain corresponding results for **Z**/p manifolds. The following mod p characteristic numbers formula is the analogue of the formula for S^1 manifolds given in §8 of [1].

THEOREM 2. *Suppose M is a unitary* **Z**/p *manifold of dimension* $2n$ *and suppose that f is any symmetric homogeneous polynomial in n variables of degree less than or equal to n, then*

$$\sum\{f(y_1, y_2, \ldots, y_d, t_1 + z_1, t_2 + z_2, \ldots, t_{n-d} + z_{n-d})\prod(t_j + z_j)^{-1}\}[F]$$

$$= f(w_1, w_2, \ldots, w_n)[M] \bmod p$$

where the sum is taken over the components F of the fixed point set. The t_i are the "rotation numbers", and elementary symmetric functions of y_i, z_i and w_i respectively, give the chern classes of F, normal bundle of F and M respectively (dim $F = 2d$).

Such a formula was known [2] but in the case that each component of the fixed point set has dimension less than $2(p-1)$, we require no such restrictions.

Let s_n be the polynomial defined by $s_n(x_1, x_2, \ldots, x_n) = x_1^n + x_2^n + \cdots + x_n^n$; then we can say more.

THEOREM 3. *Suppose M is a* **Z**/p *manifold of dimension* $2n$, *then*

$$\sum\{s_n(y_1, y_2, \ldots, y_d, t_1 + z_1, t_2 + z_2, \ldots, t_{n-d} + z_{n-d})\prod(t_j + z_j)^{-1}\}[F]$$

$$= \begin{cases} s_n[M] \bmod p^2 & \text{if } n = p^k - 1 \text{ for some } k, \\ s_n[M] \bmod p & \text{otherwise.} \end{cases}$$

Using Theorems 1 and 3 we obtain

THEOREM 4. *Let M be a* **Z**/p *manifold of dimension* $2n$ *such that*

(i) $n = 0 \bmod(p-1)$,

(ii) *either* $n = -1 \bmod p$ *or else each component of the fixed point set has trivial normal bundle in M,*

(iii) *no component of the fixed point set is of dimension* $2n$, *and*

(iv) $n \neq p^k - 1$ *for any* $k \geqslant 0$;

then M is decomposable mod p, *i.e. decomposable as an element of* $\mathfrak{A}_*/p\mathfrak{A}_*$.

As an example, **Z**/p manifolds of dimension $2n = 2(kp+1)(p-1)$, where $k > 0$ and $k \neq 1 \bmod p$, which have no component of the fixed point set of dimension $2n$ are decomposable mod p.

Finally, we mention a result whose proof uses a free $\mathfrak{A}_*$ basis for $\mathfrak{A}_*^{\mathbf{Z}/p}$.

THEOREM 5. *Let M be a $\mathbf{Z}/p$ manifold such that*

(i) *no component of M has a trivial $\mathbf{Z}/p$ action,*

(ii) *each component of the fixed point set has trivial normal bundle and*

(iii) *M has no isolated fixed points;*

then M is equivariantly decomposable modulo free $\mathbf{Z}/p$ manifolds, i.e. decomposable in $\mathfrak{A}_*^{\mathbf{Z}/p}/p\mathfrak{A}_*$.

The fact that such manifolds are decomposable in $\mathfrak{A}_*/p\mathfrak{A}_*$ follows quite easily from Theorem 3.

All the above results have obvious analogues in the oriented case so long as p is an odd prime.

Details of proofs will appear elsewhere.

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