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A WEINER-LIKE CONDITION FOR QUASILINEAR PARABOLIC EQUATIONS

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I consider the following parabolic equation

$$u_t = \operatorname{div} A(x, t, u, u_x) + B(x, t, u, u_x) \tag{1}$$

where A, B are respectively, vector and scalar valued measurable functions satisfying the structure conditions

$$\begin{gathered} |A(x, t, u, p)| \leqslant a_1 |p| + a_2 |u| + a_3, \quad |B(x, t, u, p)| \leqslant b_1 |p| + b_2^2 |u| + b_3^2, \\ p \cdot A(x, t, u, p) \geqslant c_1 |p|^2 - c_2^2 |u|^2 - c_3^2, \end{gathered} \tag{2}$$

where a_1, c_1 are positive constants, and all of the remaining coefficients a_i, b_i, c_i are in $L^{p,q}$ for some pair of numbers (p, q) satisfying $p \geqslant 2/(1-\theta)$; $n/p + 2/q \leqslant 1 - \theta$, where θ is a positive constant, $0 < \theta < 1$. This is precisely the equation studied by Aronson and Serrin [1] and is very similar to that studied by Trudinger [7].

We consider weak solutions from the class V^2 in cylinders $Q = \Omega \times (0, T)$ where $\Omega \subset \mathbf{R}^n$ is a bounded domain. $V^2(Q)$ is defined to be the space of measurable functions u which have finite norm

$$\|u\|_{V^2(Q)} = \operatorname*{ess\,sup}_{0<t<T} \left\{ \int_\Omega |u(x, t)|^2 \, dx \right\}^{1/2} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(Q)}$$

where $\{\partial u/\partial x_i\}_{i=1,\dots,n}$ are the weak (i.e. distributional) derivatives of u. We define $V_0^2(Q)$ to be the colsure in $\|\cdot\|_{V^2(Q)}$, of functions in $C^\infty(Q)$ which vanish in a neighborhood of the parabolic boundary $\partial_p Q = \overline{\Omega} \cup \{\partial\Omega \times [0, T]\}$. We say that $u \in V^2(Q)$ is a weak solution to (1) if $\int \varphi_t u - \varphi_x \cdot A(x, t, u, u_x) + \varphi B(x, t, u, u_x) = 0$ for every function $\varphi \in C_c^\infty(Q)$.

The Maximum Principle for such equations (Aronson and Serrin [1, Theorem 1], can be generalized to the notion of weak boundary values as follows.

THEOREM . *If $u \in V_0^2(Q)$ is a weak solution to* (1) *then almost everywhere in Q we have*

$$|u(x, t)| \leqslant C(\|b_3\|_{p,q}^2 + \|c_3\|_{p,q})$$

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where $C = C(T, |\Omega|, n, \theta, \|b_1\|, \|b_2\|, c_1, \|c_2\|)$.

We employ the familiar Bessel capacity $B_{1,2}$ on $\mathbf{R}^n$ (see Meyers [5]) and introduce a new capacity VC defined on $\mathbf{R}^{n+1}$ by

$$VC(A) = \operatorname{Inf}\{\|u\|_{V^2(\mathbf{R}^{n+1})}: u \geqslant 0 \text{ and } A \subset \operatorname{Int}\{(x, t): u(x, t) \geqslant 1\}\}$$

for any set $A \subset \mathbf{R}^{n+1}$. VC is an outer measure on $\mathbf{R}^{n+1}$. These capacities are employed in the following results.

THEOREM. *If* $u \in V_0^2(Q)$ *is a weak solution of* (1), *then*

$$\operatorname*{limit}_{(x,t)\to(x_0,t_0);(x,t)\in Q} u(x, t) = 0$$

for VC almost every point $(x_0, t_0) \in \partial_p Q$.

THEOREM. *Suppose* $n > 2$ *and the structure coefficients* a_i, b_i, c_i *in* (2) *are all positive constants. Suppose also that* $x_0 \in \partial\Omega$ *has the property that the* $B_{1,2}$ *upper capacitary density of* $\widetilde{\Omega}$ *is positive at* x_0, *that is*

$$\limsup_{r\to 0} \frac{B_{1,2}(B(x_0, r) \cap \widetilde{\Omega})}{B_{1,2}(B(x_0, r))} > 0.$$

If $u \in V_0^2(Q)$ *is a weak solution of* (1), *then* $\operatorname{limit}_{(x,t)\to(x_0,t_0);(x,t)\in Q} u(x, t) = 0$ *for every* $t_0 \in (0, T)$.

The notion of limit employed is, of course, the essential limit. That is, u may need to be redefined on a set of zero measure.

The last result is modeled after a similar result for elliptic equations appearing in [3]. It gives a Weiner-like geometric condition on the base region Ω which implies continuity of a weak solution at all points on the lateral boundary of the cylinder directly "above" the boundary point x_0. Proofs of all results appear in [2].

BIBLIOGRAPHY

1. D. Aronson and J. Serrin, *Local behavior of solutions of quasilinear parabolic equations*, Arch. Rational Mech. Anal. **25** (1967), 81–122. MR **39** #5952.

2. Daniel J. Deignan, *Boundary regularity of weak solutions to a quasilinear parabolic equation*, Doctoral Dissertation, Indiana University, 1974.

3. R. Gariepy and W. P. Ziemer, *Behavior at the boundary of solutions of quasilinear elliptic equations*, Arch. Rational Mech. Anal. **56** (1974/75), 372–384. MR **50** #7807.

4. O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, "Nauka", Moscow, 1967; English transl., Transl. Math. Monographs, vol. 23, Amer. Math. Soc., Providence, R. I., 1968. MR **39** #3159a, b.

5. N. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand. **26** (1970), 255–292. MR **43** #3474.

6. J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134; Errata, Ibid. **20** (1967), 231–236. MR **28** #2357; **34** #3121.

7. N. Trudinger, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math. **21** (1968), 205–226. MR **37** #1758.

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